Filter Design Techniques

• Goal:

\[ \text{Determine } H(z) \text{ or } h(n) \text{ such that } H(e^{j\omega}) \text{ meets required specifications} \]

• Typical frequency responses:

Lowpass, bandpass, highpass, differentiator

• Typical specifications:

Passband cutoff, stopband cutoff, passband errors, stopband errors

• Two major categories:

**IIR Filter**: “parameter efficient” but “not linear phase”

**FIR Filter**: “ideal linear phase” but “not parameter efficient”.
IIR Filter Design

• General approach:

  *design a digital filter from an analog filter*

• Typical analog filters:

  Butterworth filter: \( |H_c(j\Omega)|^2 = \frac{1}{1 + (j\Omega / j\Omega_c)^{2N}} \)

  *Maximal flat in passband*
  *Monotonic decreasing*
  *Transition sharpens as \( N \) increases*

  What is the transfer function?
  Where are the poles?

  Chebyshev filter (I): \( |H_c(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 V_N^2 (\Omega / \Omega_c)} \)

  \( V_N(x) = \cos(N \cos^{-1} x) \)
  \( V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x) \)

  *Equiripple in passband*
  *Monotonic in stopband*
Chebyshev filter (II):

\[ |H_c(j\Omega)|^2 = \frac{1}{1 + \left[ \epsilon^2 V_N^2 \left( \Omega_c / \Omega \right) \right]^{-1}} \]

*Equiripple in stopband*

*Monotonic in passband*

Elliptic filter:  

\[ |H_c(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 U_N^2(\Omega)} \]

\[ U_N(\Omega): \text{Jacobian elliptic function} \]

*equiripple in both passband and stopband*

*most efficient in parameters among all*

- Frequency transformations in S-domain:

  Lowpass to lowpass:  
  \[ s \rightarrow \frac{\Omega_p}{\Omega_p} s \]

  Lowpass to highpass:  
  \[ s \rightarrow \frac{\Omega_p^2 \Omega_p'}{s} \]

  Lowpass to bandpass:  
  \[ s \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \]
Lowpass to bandstop: \( s \to \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u} \)

- Design techniques (from analog to digital):
  
  **Impulse invariance:** \( h(n) = Th_c(nT) \)

  **Bilinear transformation:** \( s = \frac{2 \left(1 - z^{-1}\right)}{T \left(1 + z^{-1}\right)} \)

- **Impulse invariance:**

  Let \( h(n) = Th_c(nT) \)

  Then \( H(e^{j\omega}) = H_c\left(\frac{j\omega}{T}\right) \) for \( |\omega| \leq \pi \)

  **Assumption:** Nyquist rate must be satisfied.

  **Lowpass filter is the most proper choice.**

- Some key equations:

\[
H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}
\]

\[
h_c(t) = u_c(t) \sum_{k=1}^{N} A_k e^{s_k t}
\]

\[
h(n) = Tu(n) \sum_{k=1}^{N} A_k e^{s_k Tn}
\]
\[
H(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{s_k T} z^{-1}}
\]

A pole at \( s_k \) in the \( S \)-plane is mapped to \( e^{s_k T} \) in the \( Z \)-plane.

The same rule does NOT apply to the zeros!

- The design procedure: see Example 7.2 (p.446).

Step 1: determine the specifications in the frequency domain with \( T=1 \) without loss of generality;

Step 2: determine the parameters of the magnitude frequency response function of an analog filter (e.g., Butterworth);

Step 3: Determine the (stable) poles of the \( S \)-domain transfer function;

Step 4: Determine the fractional expansion of the \( S \)-domain transfer function;

Step 5: Determine the \( Z \)-domain transfer function in the fractional form; and then simplify.
• Bilinear transformation:

\[
\frac{s}{T} = \frac{2(1 - z^{-1})}{1 + 2^{-1}z^{-1}} \quad \text{or} \quad z = \frac{1 + (T/2)s}{1 - (T/2)s}
\]

where \( T \) can be any value.

*Left half plane of \( S \) is mapped onto inside the unit circle of \( Z \)*

*The right half plane of \( S \) is mapped onto outside the unit circle of \( Z \)*

*The imaginary axis of \( S \) is mapped onto the unit circle of \( Z \)*

\[ i.e., \quad \Omega = \frac{2}{T} \tan(\omega/2) \]

*a nonlinear mapping (see Figures 7.6-7.8)*

• The design procedure: *see Example 7.3 (p.454)*

*Step 1:* determine the specifications in the frequency domain with \( T = 1 \);

*Step 2:* determine the parameters of the magnitude frequency response function of an analog filter (e.g., Butterworth);
Step 3: Determine the (stable) poles of the S-domain transfer function;

Step 4: Determine the S-domain transfer function;

Step 5: Determine the Z-domain transfer function via the bilinear transform.

- Frequency transformations in the Z-domain:

  **Lowpass to lowpass:** \[ z^{-1} \rightarrow \frac{z^{-1} - a}{1 - az^{-1}} \]

  **Lowpass to highpass:** \[ z^{-1} \rightarrow -\frac{z^{-1} + a}{1 + az^{-1}} \]

  **Lowpass to bandpass:** \[ z^{-1} \rightarrow -\frac{z^{-2} - a_1z^{-1} + a_2}{a_2z^{-2} - a_1z^{-1} + 1} \]

  **Lowpass to bandstop:** \[ z^{-1} \rightarrow \frac{z^{-2} - a_1z^{-1} + a_2}{a_2z^{-2} - a_1z^{-1} + 1} \]

  *Where the values of a’s are chosen to meet the required cutoff frequencies (for details, see Digital Signal Processing, J. G. Proakis, p.649)*