Properties of LTI Systems

- Commutative:

$$x(n) * h(n) = h(n) * x(n)$$

- Linearity:

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

- Cascade connection:

$$\begin{aligned} y(n) &= [x(n) * h_1(n)] * h_2(n) \\ &= x(n) * [h_1(n) * h_2(n)] \end{aligned}$$

Can you sketch the diagram?

- Parallel connection:

$$\begin{aligned} y(n) &= [x(n) * h_1(n)] + [x(n) * h_2(n)] \\ &= x(n) * [h_1(n) + h_2(n)] \end{aligned}$$

Can you sketch the diagram?

- Stable LTI system:

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

sufficient and necessary

- Causal LTI system:

$$h(n) = 0 \text{ for } n < 0$$

sufficient and necessary

Linear Constant-Coefficient Difference Equation

- The general form:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$$

- Recursive computation:

$$y(n) = -\frac{1}{a_0}\sum_{k=1}^{N} a_k y(n-k) + \frac{1}{a_0}\sum_{m=0}^{M} b_m x(n-m)$$

- Accumulator:

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

or equivalently,

$$y(n) - y(n-1) = x(n)$$

Can you sketch the diagram?

- Moving average (MA):

$$y(n) = \sum_{m=0}^{M} b_m x(n-m)$$

where b_m's are the weights

- Auto regression (AR):

$$\sum_{k=0}^{N} a_k y(n-k) = x(n)$$

- Homogeneous equation:

$$\sum_{k=0}^{N} a_k y(n-k) = 0$$

where the input is *zero*!

The output *may not be zero* as

$$y(n) = \sum_{m=1}^{N} A_m z_m{}^n$$

$$\text{if } \sum_{k=1}^{N} a_k z_m{}^{-k} = 0$$

The output *is zero* if

$$y(-1) = y(-2) = \ldots = y(-N) = 0$$

- The ARMA system:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$$

is *linear, time-invariant, and causal* if

$$y(-1) = y(-2) = ... = y(-N) = 0$$

zero initial conditions

- With the zero initial conditions, the ARMA system can be uniquely represented by its impulse response $h(n)$, $0 < n < \infty$.

Frequency-Domain Representation of LTI Systems

- Eigenfunction:

  If $x(n) = \exp(j\omega n)$, $-\infty < n < \infty$, then,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)e^{j\omega n - j\omega k}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$= e^{j\omega n} H(e^{j\omega})$$

  Hence, $\exp(j\omega n)$ is called the *eigenfunction*, and $H(j\omega)$ is called the *frequency response*.

- Generally, we can write

$$x(n) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\omega_k n}$$

  Then, by linearity,

$$y(n) = \sum_{k=-\infty}^{\infty} \alpha_k H(e^{j\omega_k}) e^{j\omega_k n}$$

- Properties of frequency response:

$$H(j\omega) \text{ is complex}$$

$$H(j\omega) \text{ is periodic}$$

- Ideal delay:

$$y(n) = x(n - n_0)$$

$$h(n) = \delta(n - n_0)$$

$$H(j\omega) = e^{-j\omega n_0}$$

- Ideal frequency-selective filters:

$$|H(j\omega)| = \begin{cases} 1 & \omega \in \Omega_0 \\ 0 & otherwise \end{cases}$$