Filter Design Techniques

- Goal:

  Determine H(z) or h(n) such that $H(e^{j\omega})$ *meets required specifications*

- Typical frequency responses:

  Lowpass, bandpass, highpass, differentiator

- Typical specifications:

  Passband cutoff, stopband cutoff, passband errors, stopband errors

- Two major categories:

  IIR Filter: "parameter efficient" but "not linear phase"

  FIR Filter: "ideal linear phase" but "not parameter efficient".

IIR Filter Design

- General approach:

design a digital filter from an analog filter

- Typical analog filters:

Butterworth filter: $$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\left(j\Omega / j\Omega_c\right)^{2N}}$$

Maximal flat in passband

Monotonic descreasing

Transition sharpens as N increases

What is the transfer function?

Where are the poles?

Chebyshev filter (I): $$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 V_N{}^2(\Omega / \Omega_c)}$$

$$V_N(x) = \cos\left(N\cos^{-1} x\right)$$

$$V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x)$$

Equiripple in passband

Monotonic in stopband

Chebyshev filter (II):

$$|H_c(j\Omega)|^2 = \frac{1}{1+\left[\varepsilon^2 V_N{}^2(\Omega_c/\Omega)\right]^{-1}}$$

Equiripple in stopband

Monotonic in passband

Elliptic filter: $|H_c(j\Omega)|^2 = \dfrac{1}{1+\varepsilon^2 U_N{}^2(\Omega)}$

$U_N(\Omega)$: Jacobian elliptic function

equiripple in both passband and stopband

most efficient in parameters among all

- Frequency transformations in S-domain:

Lowpass to lowpass: $s \to \dfrac{\Omega_p}{\Omega_p} s$

Lowpass to highpass: $s \to \dfrac{\Omega_p \Omega_p{}'}{s}$

Lowpass to bandpass: $s \to \Omega_p \dfrac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$

Lowpass to bandstop: $s \rightarrow \Omega_p \dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$

- Design techniques (from analog to digital):

Impulse invariance: $h(n) = T h_c(nT)$

Bilinear transformation: $s = \dfrac{2}{T}\dfrac{1 - z^{-1}}{1 + z^{-1}}$

- Impulse invariance:

Let $h(n) = T h_c(nT)$

Then $H\left(e^{j\omega}\right) = H_c\left(j\dfrac{\omega}{T}\right)$ for $|\omega| \le \pi$

Assumption: Nyquist rate must be satisfied.

Lowpass filter is the most proper choice.

- Some key equations:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$h_c(t) = u_c(t) \sum_{k=1}^{N} A_k e^{s_k t}$$

$$h(n) = T u(n) \sum_{k=1}^{N} A_k e^{s_k T n}$$

$$H(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{s_k T} z^{-1}}$$

A pole at s_k in the S-plane is mapped to $e^{s_k T}$ in the Z-plane.

The same rule does NOT apply to the zeros!

- The design procedure: *see Example 7.2 (p.446)*.

  Step 1: determine the specifications in the frequency domain with T=1 without loss of generality;

  Step 2: determine the parameters of the magnitude frequency response function of an analog filter (e.g., Butterworth);

  Step 3: Determine the (stable) poles of the S-domain transfer function;

  Step 4: Determine the fractional expansion of the S-domain transfer function;

  Step 5: Determine the Z-domain transfer function in the fractional form; and then simplify.

- Bilinear transformation:

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \text{ or } z = \frac{1+(T/2)s}{1-(T/2)s}$$

where T can be any value.

Left half plane of S is mapped onto inside the unit circle of Z

The right half plane of S is mapped onto outside the unit circle of Z

The imaginary axis of S is mapped onto the unit circle of Z

i.e., $\Omega = (2/T)\tan(\omega/2)$

a nonlinear mapping (see Figures 7.6-7.8)

- The design procedure: *see Example 7.3 (p.454)*

  Step 1: determine the specifications in the frequency domain with T=1;

  Step 2: determine the parameters of the magnitude frequency response function of an analog filter (e.g., Butterworth);

Step 3: Determine the (stable) poles of the S-domain transfer function;

Step 4: Determine the S-domain transfer function;

Step 5: Determine the Z-domain transfer function via the bilinear transform.

- Frequency transformations in the Z-domain:

$$\textit{Lowpass to lowpass: } z^{-1} \to \frac{z^{-1} - a}{1 - az^{-1}}$$

$$\textit{Lowpass to highpass: } z^{-1} \to -\frac{z^{-1} + a}{1 + az^{-1}}$$

$$\textit{Lowpass to bandpass: } z^{-1} \to -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

$$\textit{Lowpass to bandstop: } z^{-1} \to \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

Where the values of a's are chosen to meet the required cutoff frequencies (for details, see Digital Signal Processing, J. G. Proakis, p.649)