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where \( a_i \) is the \( i \)-th complex amplitude (none zero); \((y_i, z_i) = (e^{j\omega_i}, e^{j\omega_i})\) defines the \( i \)-th 2-D frequency \((\omega_i, \omega_i)\); \( N_z \) is the number of the 2-D sinusoids. Naturally, \((y_i, z_i)\) should be distinct. Let \( d_y \) and \( d_z \) be the numbers of distinct poles in \( \{y_i\} \) and \( \{z_i\} \), respectively. Let \( m_y \) and \( m_z \) be the maximum multiplicity in \( \{y_i\} \) and the maximum multiplicity in \( \{z_i\} \), respectively.

It follows from the definition that \( m_y \leq d_y \) and \( m_z \leq d_z \).

The block Hankel matrix of \( x(m, n) \) is defined as

\[
X_e = \begin{bmatrix}
X_0 & X_1 & \cdots & X_{M-K} \\
X_1 & X_2 & \cdots & X_{M-K+1} \\
\vdots & \vdots & \ddots & \vdots \\
X_{K-1} & X_K & \cdots & X_{M-1}
\end{bmatrix}
\]

where for \( m = 0, 1, \ldots, M-1 \),

\[
X_{em} = \begin{bmatrix}
x(m, 0) & x(m, 1) & \cdots & x(m, N-L) \\
x(m, 1) & x(m, 2) & \cdots & x(m, N-L+1) \\
\vdots & \vdots & \ddots & \vdots \\
x(m, L-1) & x(m, L) & \cdots & x(m, N-1)
\end{bmatrix}
\]

in which \( K \) and \( L \) are called the window-size parameters.

Based on (1), the block Hankel matrix \( X_e \) can be decomposed as follows. We define

\[
V(e, r) = \begin{bmatrix}
ev_0 & ev_0 & \cdots & ev_{N_y} \\
ev_1 & ev_1 & \cdots & ev_{N_y} \\
\vdots & \vdots & \ddots & \vdots \\
ev_{N_y} & ev_{N_y} & \cdots & ev_{N_y}
\end{bmatrix}
\]

where \( e = (v_0, v_1, \ldots, v_{N_y})^T \) is a vector. Then we can write (also shown in [2])

\[
X_e = X_e(K, L) = E_iA_dE_r
\]

where

\[
E_i = E_i(K, L) = \begin{bmatrix}
V(z, L) \\
V(z, L)Y_{d_2} \\
\vdots \\
V(z, L)Y_{d_2}^{K-1}
\end{bmatrix}_{KL \times N_z}
\]

and

\[
A_d = \begin{bmatrix}
d_1 & 0 \\
0 & \vdots \\
0 & d_2
\end{bmatrix}_{N_z \times N_z}
\]

It is clear from (2) that \( \text{rank}(X_e) \leq N_z \). What we need to address next is the conditions that should be satisfied by the window-size parameters \( K \) and \( L \) such that \( \text{rank}(X_e) \) is equal to the desired value \( N_z \). The general results are shown in Fig. 1, and the detailed derivations are given in the next section.

### III. The Conditions on the Window Size

Note that the matrix \( E_i \) takes a form identical to that of the following matrix:

\[
O_k = O_k(C, A) = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{k-1}
\end{bmatrix}
\]

where \( k \geq 1 \), and \( A \) and \( C \) are \( n \times n \) and \( m \times n \) matrices. When \( k = n \), \( O_n \) is known as the observability matrix of the pair \((C, A)\). Associated with the matrix \( O_k \), the following two lemmas will be useful.

**Lemma 1:** Let \( n_1 \) be the order of the minimum polynomial of \( A \). Then for \( n' \geq n_1 \), \( \text{rank}(O_{n'}) = n \) iff \( \text{rank}(\left[ A^{n'} - \lambda I \right]) = n \) for all eigenvalues \( \lambda \) of \( A \).

**Proof:** This lemma is a generalization of the Theorem 9 in [1] (p. 240).

**Lemma 2:** If \( \text{rank}(\left[ A^n - \lambda I \right]) < n \) for some eigenvalue of \( A \), then \( \text{rank}(O_k) < n \) for all \( k \geq 1 \).

**Proof:** Let \( \lambda \) be the eigenvalue of \( A \) such that \( \text{rank}(\left[ A^n - \lambda I \right]) < n \). Then there is a vector \( x \neq 0 \) such that \( \lambda x = Ax \) and \( Cx = 0 \). So \( CA^kx = 0 \) for all \( k \geq 1 \). Hence, \( O_kx = 0 \) for all \( k \geq 1 \). Therefore, \( \text{rank}(O_k) < n \) for all \( k \geq 1 \).

Now we are ready to provide the following.

**Theorem 1:**

a) If \( K \geq d_y \), then \( \text{rank}(E_i(K, L)) = N_z \) iff \( L \geq m_z \).

b) If \( L \geq d_z + 1 \), then \( \text{rank}(E_r(K, L)) = N_z \) iff \( K \geq m_z \). If \( L \leq N - d_z + 1 \), then \( \text{rank}(E_r(K, L)) = N_z \) iff \( K \leq M - m_z + 1 \).

c) \( \text{rank}(X_e(K, L)) = N_z \) if the conditions in a) and b) are met.

**Proof:** First, we consider \( E_i(K, L) \). Let \((C, A) = [V(z, L), Y_d]\). Since \( A \) is a diagonal matrix, all eigenvalues of \( A \) are the elements on the diagonal. Let \( \lambda = y_i \) be an element in \( \{y_i\} \) with a multiplicity \( m_z \), then we can transform, by a simple column permutation as follows:

\[
\begin{bmatrix}
M - A \\
C
\end{bmatrix}
\begin{bmatrix}
D_1 & 0 \\
0 & \vdots \\
0 & D_2
\end{bmatrix}
\]

where...

[Fig. 1. Regions of window size parameters. For region GI, the block Hankel matrix is of a rank less than the desired. Region G0, is the uncertain region where the block Hankel matrix may or may not have its rank equal to the desired. M and N define the size of the original data set. m_y and m_z denote the maximum multiplicities of poles in the first and second dimensions, respectively. d_y and d_z denote the numbers of distinct poles in the first and second dimensions, respectively. The sufficient condition given in [2] is a subset of GI.]

...
and, hence, we have
\[
\text{rank}\left(\begin{bmatrix} M - A \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} D_1 & 0 \\ 0 & V_1 V_2^H \end{bmatrix}\right)
\]
\[
= \text{rank}(D_1) + \text{rank}(V_2)
\]
where \( D_1 = \text{diag}(y_1 - y_1', \ldots, y_n - y_n') \), \( V_2 = V' = \begin{bmatrix} z_1', \ldots, z_m' \end{bmatrix}^T, I \), none of \( \{y_1, \ldots, y_n\} \) is equal to \( y_i \), and the polynomials in \( \{z_1', \ldots, z_m'\} \subset \{\zeta_1, \ldots, \zeta_N\} \) are distinct. So the rank of \( D_1 \) is \( N_s - m \). The rank of \( V_2 \) is \( m \) iff \( L \geq m \). Hence, \( \text{rank}\left(\begin{bmatrix} M - A \end{bmatrix}\right) = N_s \) for \( \lambda = y_1 \) iff \( L \geq m \). Therefore, \( \text{rank}\left(\begin{bmatrix} M - A \end{bmatrix}\right) = N_s \) for all \( \lambda \in \{y\} \) iff \( L \geq m \). Since \( A \) has \( d_y \) distinct diagonal elements, the order of the minimum polynomial of \( A \) is equal to \( d_y \). So, by Lemma 1, for \( K \geq d_y \),
\[
\text{rank}(E_{\lambda}(K, L)) = N_s \text{ iff } L \geq m_y.
\]
We can symmetrically get the other half conclusion of (a) by using the fact that there exists a permutation matrix \( P \) [2] such that
\[
P E_{\lambda} = \begin{bmatrix} V(y, K) & Z_d \\ V(y, K)Z_d & \vdots & \vdots \end{bmatrix} = O_1[V(y, K), Z_d] \quad (5)
\]
where \( y = (y_1, \ldots, y_N)^T \) and \( Z_d = \text{diag}(z_1', \ldots, z_N') \).

Part (b) follows readily from part (a) since \( E_{\lambda} = E_{\lambda}^T (M - K + 1, N - L + 1) \).

From (2), we know that \( \text{rank}(X_{\lambda}(K)) = N_s \) if \( \text{rank}(E_{\lambda}) = N_s \). Combining (a) and (b), we have (c). Q.E.D.

**Theorem 2:**
1) \( \text{rank}(E_{\lambda}(K, L)) = N_s \) if \( L < m_y \) or \( K < m_x \).
2) \( \text{rank}(E_{\lambda}(K, L)) < N_s \) if \( L > N - m_y + 1 \) or \( K > M - m_x + 1 \).
3) \( \text{rank}(X_{\lambda}(K, L)) < N_s \) if \( L < m_y \) or \( K < m_x \) or \( K > M - m_x + 1 \).

**Proof:** Following the first paragraph of the proof for Theorem 1, one can show that if \( \lambda = y_1 \) that has the maximum multiplicity \( m_y \), then \( \text{rank}(V_2) < m_y \) if \( L < m_y \) and hence \( \text{rank}(\begin{bmatrix} M - A \end{bmatrix}) \) is empty when \( L < m_y \). Therefore, by Lemma 2, \( \text{rank}(E_{\lambda}(K, L)) = N_s \) if \( L < m_y \). Other proofs can be done similarly. Q.E.D.

By Theorems 1 and 2, the whole window size set \( G = \{K, L\} : 1 \leq K \leq M, 1 \leq L \leq N \} \) is divided into three sets \( G = G_0 \cup G_1 \cup G_2 \) (see Fig. 1). For the sets \( G_0 \) and \( G_1 \), we have definite answers: i.e., \( \text{rank}(X_{\lambda}(K, L)) = N_s \) when \( (K, L) \in G_0 \), and \( \text{rank}(X_{\lambda}(K, L)) = N_s \) when \( (K, L) \in G_1 \). But for \( G_2 \), the answer is uncertain depending on the distribution of the signal poles. In other words, \( \text{rank}(X_{\lambda}(K, L)) \) may be less than or equal to \( N_s \) for \( (K, L) \in G_2 \). It should be noted that the sufficient condition given in [2], i.e., \( N_s \leq K \leq M - N_y + 1 \) and \( N_s \leq L \leq N - N_y + 1 \), is a subset of \( G_1 \).

When \( m_y = d_y \) and \( m_x = d_x \), the set \( G_a \) is empty. From Theorems 1 and 2, we have the following sufficient and necessary constraint on the window size under which the rank of the enhanced data matrix is equal to the desired number.

**Corollary 1:** If \( m_y = d_y \) or \( m_x = d_x \), then
\[
\text{rank}(X_{\lambda}(K, L)) = N_s \text{ iff } d_x \leq K \leq M - d_x + 1,
\]
\[
\text{or } d_y \leq L \leq N - d_y + 1.
\]
Note that the condition of Corollary 1 is satisfied when all 2-D frequencies are scattered on a rectangular grid (not necessarily uniform) and at least one straight line in each dimension on the grid is fully occupied by 2-D frequencies, which is illustrated in Fig. 2.

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**High-Speed Systolic Ladder Structures for Multidimensional Recursive Digital Filters**

Xiaojian Liu and Leonard T. Bruton

**Abstract—** We propose a multilevel approach for designing high-speed systolic ladder structures for multidimensional (MD) recursive digital filters. Based on appropriately selected 1-D filter structures for each filter dimension (or level), a large variety of MD systolic filter structures may be derived. In particular, we introduce a new 1-D filter structure that proves the most suitable structure in terms of a systolic ladder implementation, because it leads to MD ladder filter structures possessing such important properties as the shortest critical path (for filters without order augmentation), the canonical number of high-level storage registers (e.g., line and frame registers of images), and local interconnectivity.

I. INTRODUCTION

High-speed multidimensional (MD) digital filtering is very useful for real-time video signal processing such as video image coding, bandwidth compression, sampling rate conversion and the enhancement of television signals. In this contribution, we are concerned

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