Leader–follower swarm tracking for networked Lagrange systems

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\textbf{ABSTRACT}

In this paper, swarm tracking problems with group dispersion and cohesion behaviors are discussed for a group of Lagrange systems. The agent group is separated into two subgroups. One is called the leader group, whose members are encapsulated with the desired generalized coordinates and generalized coordinate derivatives. The other one, referred to as the follower group, is guided by the leader group. The objective is to guarantee distributed tracking of generalized coordinate derivatives for the followers and to drive the generalized coordinates of the followers close to the convex hull formed by those of the leaders. Both the case of constant leaders’ generalized coordinate derivatives and the case of time-varying leaders’ generalized coordinate derivatives are considered. The proposed control algorithms are shown to achieve velocity matching, connectivity maintenance and collision avoidance. In addition, the sum of the steady-state distances between the followers and the convex hull formed by the leaders is shown to be bounded and the bound is explicitly given. Simulation results are presented to validate the effectiveness of theoretical conclusions.

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1. Introduction

Coordination of networked Lagrange systems has been receiving significant attention recently. This research interest is aroused in part by the rapid development of distributed control of multi-agent systems. We refer the readers to [1] and [2] for an overview of these research efforts. Traditionally, the system model is often simplified to that of a single-integrator kinematics or a double-integrator dynamics to highlight the interactions among different agents. But this simplification imposes an obvious limitation on the model’s abilities to represent the real physical objects. On the other hand, a Lagrange model is often used to describe mechanical systems, such as mobile robots, autonomous vehicles, robotic manipulators, and rigid bodies. Indeed, coordination of networked Lagrange systems has numerous practical applications. One typical example of such applications is the relative attitude keeping problem in the context of deep space interferometry [3–5].

Much effort has been made toward coordination problems of networked Lagrange systems. For instance, with the attitude kinematic and dynamic equations transformed into the Lagrange model, the author of [6] solves the leader–follower cooperative attitude synchronization problem where there exists a time-varying leader. Global exponential stability and various communication topologies are considered in [7] and [8] for consensus tracking of a group of Lagrange systems, where the nonlinear contraction analysis is introduced. The cases of actuator saturation and unavailability of measurements of generalized coordinate derivatives are discussed in [9]. Communication delays and dynamic topologies are considered in [10], where collision avoidance behavior is highlighted. An adaptive approach is introduced in [11] to compensate for the unknown parameters in the Lagrange dynamic models. In addition, Ref. [12] takes into consideration delays, limited data rates and bounded disturbance input in the design of the control law. An ultimate boundedness result instead of an absolute tracking result is obtained.

Group dispersion and cohesion behaviors are often very important for coordination of multi-agent systems. Group dispersion is to ensure minimum safety distance between different agents and group cohesion is to maintain the connectivity once two agents are connected. A variable structure approach is taken in [13,14] to guarantee cooperative swarm tracking for a group of agents with or without a leader. Ref. [15] extends this result to the case of a general directed communication topology. In [16], the framework
of a leaderless and a leader-following flocking is given, where three behaviors, i.e., velocity matching, cohesion, and collision avoidance are established. Connectivity maintenance approaches are proposed in [17] and [18], where a bounded or an unbounded input function is introduced. Most of the existing works on flocking with a leader relies on the strict assumption that all the followers have access to the leader’s information. In contrast, a variable structure approach is developed in [19] to address a swarm tracking problem with reduced interaction. For multiple Lagrange systems, coordination and collision avoidance are studied in [20], where the case of cooperative regulation is considered. The authors of [21] designed a so-called region-based shape control algorithm to force a group of mobile robots modeled by Lagrange dynamics to move into a desired region while maintaining a minimum distance among themselves. However, this algorithm relies on the strict assumptions that the minimum distance be small enough and all the followers have access to the information of the desired region.

Although there are many results on coordination of multi-agent systems, we note that the existing research often considers a leaderless or a one-leader case. The case of multiple leaders, where the leaders form a cohesive inclusion and the followers are guided by the leaders, is also of practical value, for example, in the analysis of collective behaviors of biological groups or in a rescue mission in a disaster area. Here, the term cohesive inclusion means a rigidly enclosed space formed by the leaders. With multiple leaders, the system robustness can be improved and the design flexibility can be increased. The concept of multi-leader was proposed in [22], which also gives a containment control algorithm to solve the multi-leader problem. Here, “containment”, also referred to as “cohesive inclusion”, means the containment of the leaders. Refs. [23,24] extend the results given in [22] to the case of switching communication topologies. In [22–24], the system model is simplified to that of a single-integrator kinematics. Ref. [25] extends this simplification to the case of attitude containment control for multiple rigid bodies. Finite-time attitude containment control problems are addressed in [26] for both cases of multiple stationary and dynamic leaders.

In this paper, we focus on the swarm tracking problem in the presence of multiple leaders and multiple followers. In particular, we establish the leader–follower swarm tracking framework with group dispersion and cohesion behaviors, where there exist multiple leaders and multiple followers. In our formulation, the system model is described as more realistic nonlinear Lagrange dynamics, instead of simpler single-integrator kinematics or double-integrator dynamics. The information interaction is assumed to be strictly distributed, i.e., the leaders’ information is available to only a portion of the followers. This is a rather mild assumption compared with those in the existing works, such as [14,16,21], especially when the leaders’ generalized coordinate derivatives are time varying. We further show that only a compromised result can be obtained when the group dispersion and cohesion behaviors and the containment objective are all considered together, i.e., the sum of the steady-state distances between the followers and the convex hull formed by the leaders might be bounded instead of approaching zero. In addition, we give an explicit description of the magnitude of this bound.

The remainder of this paper is organized as follows. In Section 2, we state the problem to be solved and present some relevant background materials. In Sections 3 and 4, we derive swarm tracking control algorithms for the followers when the leaders’ generalized coordinate derivatives are, respectively, constant and time-varying, where the detailed analysis is given in Appendix. The proposed control algorithms are shown to achieve velocity matching, connectivity maintenance, collision avoidance and containment boundedness. Simulation results are presented in Section 5 to validate our control laws and Section 6 contains our conclusions.

2. Background and problem statement

2.1. Lagrange dynamics

Suppose that there are n follower Lagrange systems. The dynamics of the Lagrange systems are described as

\[ M_i(q_i) \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + g_i(q_i) = \tau_i, \quad i = 1, 2, \ldots, n, \]

where \( M_i \) is the p × p inertia (symmetric) matrix, \( C_i(q_i, \dot{q}_i) \) is the Coriolis and centrifugal terms, \( g_i(q_i) \) is the vector of gravitational force, and \( \tau_i \) is the control force. Note that the dynamics of a Lagrange system satisfies the following properties.

1. There exist positive constants \( k_M, k_C, k_T, k_g \) such that \( k_M \dot{q}_i \leq M_i(q_i) \leq k_M \dot{q}_i \), \( \| C_i(q_i, \dot{q}_i) \| \leq k_C \| \dot{q}_i \| \), and \( \| g_i(q_i) \| \leq k_g \).

2. \( M_i(q_i) - 2C_i(q_i, \dot{q}_i) \) is skew symmetric.

3. The left-hand side of the dynamics can be parameterized, i.e.,

\[ M_i(q_i)X + C_i(q_i, \dot{q}_i)Y + g_i(q_i) = Y_i(q_i, \dot{q}_i, x, y) \theta_i, \quad \forall x, y \in \mathbb{R}^p, \]

where \( Y_i \) is a \( p \times n \) matrix with a constant parameter vector \( \theta_i \) in \( \mathbb{R}^p \).

From property 3, we know that the nominal dynamics satisfy

\[ \tilde{M}_i(q_i) \ddot{\hat{q}}_i + \tilde{C}_i(q_i, \dot{\hat{q}}_i) \dot{\hat{q}}_i + \tilde{g}_i(q_i) = Y_i(q_i, \dot{\hat{q}}_i, \dot{\hat{q}}_i) \tilde{\theta}_i, \]

where \( \tilde{M}_i, \tilde{C}_i, \tilde{g}_i, \) and \( \tilde{\theta}_i \) are nominal dynamics terms. For later use, we define

\[ \psi_i(t) = M_i(q_i) \ddot{\hat{q}}_i + C_i(q_i, \dot{\hat{q}}_i) \dot{\hat{q}}_i + g_i(q_i) = Y_i(q_i, \dot{\hat{q}}_i, \dot{\hat{q}}_i) \Delta \theta_i, \]

where \( M_i(q_i) = M_i(q_i) - \tilde{M}_i(q_i), \Delta C_i(q_i, \dot{\hat{q}}_i) = \tilde{C}_i(q_i, \dot{\hat{q}}_i) - C_i(q_i, \dot{\hat{q}}_i), \Delta g_i(q_i) = g_i(q_i) - \tilde{g}_i(q_i), \) and \( \Delta \theta_i = \dot{\hat{\theta}} - \hat{\theta} \).

Suppose that in addition to the n follower agents with Lagrange dynamics, there are m leader agents with the desired generalized coordinates and generalized coordinate derivatives. Our goal here is to drive the generalized coordinate derivatives of the followers to converge to those of the leaders and to force the generalized coordinates of the followers close to the cohesive inclusion formed by both the leaders. The both of constant and time-varying leaders’ generalized coordinate derivatives will be discussed.

2.2. Graph theory

We will use graph theory to model the communication topology among agents (both followers and leaders). A directed graph \( \mathcal{G} \) consists of a pair \( (V, E) \), where \( V = \{ v_1, v_2, \ldots, v_{n+m} \} \) is a finite nonempty set of nodes and \( E \subseteq V \times V \) is a set of ordered pairs of nodes. An edge \((v_i, v_j)\) denotes that node \( v_i \) obtains information from node \( v_j \).

All neighbors of node \( v_i \) are denoted as \( N_i := \{ v_j | (v_j, v_i) \in E \} \).

An undirected graph is defined such that \( (v_i, v_j) \in E \) implies \( (v_j, v_i) \in E \). A directed path in a directed graph or an undirected path in an undirected graph is a sequence of edges of the form \( (v_1, v_2), (v_2, v_3), \ldots \).

The adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{(m+n) \times (m+n)} \) associated with the directed graph \( \mathcal{G} \) is defined such that \( a_{ij} \) is positive if \( (v_i, v_j) \in E \) and \( a_{ij} = 0 \) otherwise. For the undirected graph, we assume that \( a_{ij} = a_{ji} \). In this paper, we assume that \( a_{ji} = 0, \forall i \). The Laplacian matrix \( L = [l_{ij}] \in \mathbb{R}^{(m+n) \times (m+n)} \) associated with \( A \) is defined as \( l_{ii} = \sum_{j \neq i} a_{ij} \) and \( l_{ij} = -a_{ij} \), where \( i \neq j \).

Definition 2.1. Suppose that there exist \( m \) leader nodes and \( n \) follower nodes. Without loss of generality, we let nodes \( v_1 \) to \( v_m \)
represent the followers, and nodes $v_{n+1}$ to $v_{n+m}$ represent the leaders. The follower set and the leader set are denoted as, respectively, $F := \{ v_1, v_2, \ldots, v_n \}$ and $L := \{ v_{n+1}, v_{n+2}, \ldots, v_{n+m} \}$.

We also define the communication graphs for the generalized coordinates and the generalized coordinate derivatives, respectively. Consistent with \[10\] and \[27\], we assume that all the followers are equipped with communication units and sensing units. The sensing unit accounts for the measurements of relative generalized coordinates between different agents and the communication unit accounts for the measurements of relative generalized coordinate derivatives. In such case, we use, respectively, the sensing graph $\bar{g}^S := (V = L \cup F, \bar{E}^S)$ (or the generalized coordinate graph) and the communication graph $\bar{g}^C := (V, \bar{E}^C)$ (or the generalized coordinate derivative graph) to denote the information interaction between different agents, where $\bar{E}^S$ and $\bar{E}^C$ are defined in Definition 2.2 below.

**Definition 2.2.** The neighbors of the followers and the leaders in the generalized coordinate graph $\bar{g}^S$ are defined as $N_i^S := \{ v_j | (v_i, v_j) \in \bar{E}^S \}$, where

$$\bar{g}^S := \{ (v_i, v_j) \in V \times V \mid v_i \leftrightarrow v_j \}, \quad \forall i \in V, \forall j \in F \cup L.$$

Note that $r$ denotes the sensing radius.

**Definition 2.3.** The neighbors of the followers and the leaders in the generalized coordinate derivative graph $\bar{g}^C$ are defined as $N_i^C := \{ v_j | (v_i, v_j) \in \bar{E}^C \}$, where

$$\bar{g}^C := \{ (v_i, v_j) \in V \times V | v_i \leftrightarrow v_j \}, \quad \forall i \in F, \forall j \in F \cup L \cup L \cup L \cup L.$$

$\leftrightarrow$ denotes unordered adjacency, and $\Rightarrow$ denotes ordered adjacency.

2.3. Graph connectivity assumptions and Laplacian matrix decomposition

In this paper, the following graph connectivity assumptions will be made to guarantee the necessary information sharing within the group.

**Assumption 2.1.** For each follower, there exists at least one leader that has a path to the follower at the initial time $t = 0$ in the sensing graph $\bar{g}^S$.

**Assumption 2.2.** The graph connectivity relationship is fixed and for each follower, there exists at least one leader that has a path to the follower in the communication graph $\bar{g}^C$.

By expanding the Kronecker product, we have that

$$(\mathcal{L} \otimes I_p) \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix} = \begin{bmatrix} \mathcal{L} \otimes I_p & \mathcal{I}_d \otimes I_p \\ 0_{p\times p} & 0_{p\times p} \end{bmatrix} \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix},$$

where $\mathcal{L} \in \mathbb{R}^{n \times n}$, $\mathcal{I}_d \in \mathbb{R}^{n \times m}$, and

$$x_i = \begin{bmatrix} x_i^T_1, \ldots, x_i^T_n \end{bmatrix} \in \mathbb{R}^{p \times n}.$$
Assume that $\hat{g}^5$ satisfies Assumption 2.1 and $g^C$ satisfies Assumption 2.2. Also assume that $\|q_s(0) - q_i(0)\| > d_1$ for all $i, j \in V, i \neq j$. By using the proposed distributed control law (3) with (4)-(7) for followers’ dynamics (1), we can conclude that

1. $\mathcal{N}_i^S(0) \subseteq \mathcal{N}_i^S(t)$ for all $i \in F$ and $t \geq 0$.
2. $\dot{q}_i \rightarrow \dot{q}_d, \forall i \in V$.
3. $\|q_i(t) - q_j(t)\| > d_1$ for all $i, j \in V, i \neq j$.
4. $\lim_{t \rightarrow \infty} \varphi \leq \frac{n(n+m)}{\lambda_{\min}(T^S)} \alpha^*$ for some $\alpha^* > 0$, where $\varphi$ and $T^S$ are as defined in Definition 3.1 and Section 2.3.

**Proof.** The proof involves four parts: connectivity maintenance analysis, velocity matching analysis, group dispersion analysis, and containment boundedness analysis. In the connectivity maintenance analysis, we show that no edge in $\hat{g}^5$ will be lost for $t \geq 0$ if the initial connectivity relationship for $g^5$ satisfies Assumption 2.1. In the velocity matching analysis, we prove that the generalized coordinate derivatives for the followers will track those of the leaders. In the group dispersion analysis, the containment boundedness analysis, the group dispersion and cohesion behaviors within the group will be evaluated. The detailed proof can be found in Appendix A.3. □

**Remark 3.1.** As seen in the proof of Theorem 3.1 in Appendix A.3, the bound on the sum of the steady-state distances between the followers and the convex hull formed by the leaders is related to the sensing radius $r$, the minimum safety distance $d_1$, the cohesive radius $d_2$, the numbers of leaders and followers, $\lambda_{\min}(T^S)$, and the system initial state $U(0)$.

**Remark 3.2.** The group dispersion result readily implies collision avoidance of different agents within the group. The extension to the case of avoidance of external obstacles, as considered in [16], is important in some applications and will be one of our future research directions.

**Remark 3.3.** In this paper, we assume that the leaders’ generalized coordinate derivatives are identical. In such a situation, the formation of the leaders is fixed. The extension to the case of a time-varying leader formation is of interest and will need further consideration.

3.2. Extension to the case where the communication graph is replaced by the sensing graph

We note that the sensing graph and the communication graph are considered separately in Section 3.1. In addition, the communication graph is assumed to be connected for all time. In this section, we replace the communication graph with the sensing graph and only impose an assumption on the initial connectivity relationship. In such a case, the generalized coordinate derivative estimator is replaced by

$$\dot{\hat{v}}_i = \sum_{j=1}^{n+m} a_j(q)(\hat{v}_i - \hat{v}_j) - \delta \left( \sum_{j=1}^{n+m} a_j(q)(q_i - q_j) + \sum_{j=1}^{n+m} \frac{\partial V_y}{\partial q_j} \right), \quad i \in F,$$

$$\dot{\hat{v}}_j = \dot{\hat{q}}_d, j \in L.$$

Theorem 3.2. Assume that $\hat{g}^5$ satisfies Assumption 2.1. Also assume that $\|q_i(0) - q_j(0)\| > d_1$ for all $i, j \in V, i \neq j$. By using the proposed distributed control law (3) with (4), (5), (7), and (8) for the followers’ dynamics (1), we can conclude that

1. $\mathcal{N}_i^S(0) \subseteq \mathcal{N}_i^S(t)$ for all $i \in F$ and $t \geq 0$.
2. $\dot{q}_i \rightarrow \dot{q}_d, \forall i \in F$.
3. $\|q_i(t) - q_j(t)\| > d_1$ for all $i, j \in V, i \neq j$.
4. $\lim_{t \rightarrow \infty} \varphi \leq \frac{n(n+m)}{\lambda_{\min}(T^S)} \alpha^*$ for some $\alpha^* > 0$.

**Proof.** The proof is similar to that of Theorem 3.1. Construct the same Lyapunov function as (15). It is easy to show that $\hat{g}^5$ switches at $t_k = 1, 2, \ldots$. Then, following the same analysis as given in the proof of Theorem 3.1, we know that for $t \in [0, t_k), U(t) \leq U(0)$. By the definition of $V_y$,\n
$$\lim_{\|q_i - q_j\| \rightarrow \infty} V_y = \infty,$$

and thus no edge will be lost at time $t_k$ for $i, j \in V$. Therefore, new edges must be added for $\hat{g}^5$ at switching time $t_k$. The definition of $V_y$ also guarantees the boundedness and continuity of $U$. Therefore, we can verify that $U(t_k)$ is bounded. Similar to the aforementioned analysis, it follows that no edge will be lost for $i, j \in V$ and $t \in [t_k, t_{k+1})$. Therefore, $\mathcal{N}_i^S(0) \subseteq \mathcal{N}_i^S(t)$ for all $i \in F$ and $t \geq 0$.

The velocity matching analysis, group dispersion analysis, and containment boundedness analysis are the same as those in the proof of Theorem 3.1. □

4. Followers’ swarm tracking control when the leaders’ generalized coordinate derivatives are time-varying

In this section, $\dot{q}_i \in \mathbb{R}^p, i \in L$, is assumed to be identical and time-varying. We let $\dot{q}_i = \dot{q}_d, i \in L$.

4.1. Followers’ swarm tracking control

The control goal here is the same as the one in Section 3, i.e., to drive the generalized coordinate derivatives of the followers to converge to $\hat{q}_d$ and the generalized coordinates of the followers close to the cohesive inclusion formed by the leaders. We will use a variable structure approach instead of an adaptive control method to compensate for the model uncertainties. Thus, besides assuming that $\dot{q}_d$ and $\dot{q}_d$ are bounded, we also assume that $\Delta \theta_i, i \in F$, is bounded, where $\Delta \theta$ is defined in Section 2.1.2 The proposed control law for the followers is given by

$$\tau_i = \hat{C}\dot{q}_i + \hat{g}_i + \kappa_i - \hat{M}_i \left( \frac{d}{dt} \sum_{j=1}^{n+m} a_j(q)(q_i - q_j) \right) + \frac{d}{dt} \left( \sum_{j=1}^{n+m} \frac{\partial V_y}{\partial q_i} \right) + \kappa_i, \quad i \in F,$$

where $\hat{M}_i, \hat{C}_i$ and $\hat{g}_i$ represent $M_i(q_i, \hat{\theta}_i), C_i(q_i, \hat{q}_i, \hat{\theta}_i)$ and $g_i(q_i, \hat{\theta}_i)$ as given in Section 2.1, $V_y$ and $a_j$ are defined in Appendix A.1 and \footnotetext[2]{This is naturally satisfied for constant $\hat{\theta}_i$ and $\hat{\theta}_i$.}
Different from Section 2.1, and $K$ is any positive constant. The sliding surface is specified as

$$
s_i = \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) + \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} + \sum_{j=1}^{n+m} b_{ij}(q_i - q_j), \quad i \in F,
$$

and $\kappa_i$ is given by

$$
\kappa_i = -\mu_i Y_i \text{sgn} \left( Y_i^T \hat{M} i \left( \sum_{j=1}^{n+m} b_{ij}(s_i - s_j) \right) \right) - \rho_i \hat{M} i \text{sgn} \left( \sum_{j=1}^{n+m} b_{ij}(s_i - s_j) \right), \quad i \in F,
$$

where $Y_i$ is as defined in Section 2.1, $b_{ij}$ denotes the $(i, j)$th entry of the adjacency matrix $A^C = [b_{ij}]$ associated with $g^C$, defined in Section 2.2, $s_i = 0$, $\forall j \in L$, $\text{sgn}(x) = [\text{sgn}(x_1), \text{sgn}(x_2), \ldots, \text{sgn}(x_n)]^T$ for $x = [x_1, x_2, \ldots, x_n]^T$, with $\text{sgn}$ being the signum function, and $\mu_i$ and $\rho_i$ are positive constants, whose values are to be specified.

Applying the control law (9) to the followers’ dynamics (1) leads to

$$
\hat{M}_i \frac{dt}{dt} \left( \hat{q}_i - \hat{q}_d \right) = -\hat{M}_i \hat{q}_d - Y(q_i, \hat{q}_i, \hat{q}_d) \triangle \theta_i - \hat{M}_i L_i + \left( \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial q_i} \right) + \kappa_i, \quad i \in F.
$$

**Theorem 4.1.** Assume that $g^S$ satisfies Assumption 2.1 and $g^C$ satisfies Assumption 2.2. Also assume that $\|q_i(t) - q_j(0)\| > d_1$ for all $i, j \in F$, $i \neq j$. If $\mu_i > \|\triangle \theta_i\|_\infty$ and $\rho_i > \|\hat{q}_d\|_\infty$, $\forall i \in F$, then, by using the proposed distributed control law (9) with (10) and (11) for the followers’ dynamics (1), we can conclude that

1. $\hat{q}_i \to \hat{q}_d, \forall i \in F$.
2. $\hat{q}_d \to \text{co}(q_j, j \in L), \forall i \in F$.

**Proof.** See Appendix A.4. □

**Remark 4.1.** Different from Section 3, we do not use the generalized coordinate derivative estimator to obtain the leaders’ generalized coordinate derivatives in this section. Thus, a large amount of calculation is avoided.

4.2. Extension to containment control

In Section 4.1, the bound on the sum of the steady-state distances between the followers and the convex hull formed by the leaders might not be zero when we have other requirements on group cohesion and dispersion. In this section, group cohesion and dispersion behaviors are not considered in the control law. In such a case, we will show that the followers will converge into the convex hull formed by the leaders, i.e., the bound on the sum of the steady-state distances between the followers and the convex hull formed by the leaders will converge to zero. The proposed control law for the followers is given by

$$
\tau_i = \hat{C}_i \hat{q}_i + \hat{g}_i - \hat{M}_i \frac{dt}{dt} \left( \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) \right) - K \hat{M}_i s_i + \kappa_i, \quad i \in F,
$$

where the sliding mode is defined as

$$
s_i = \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) + \sum_{j=1}^{n+m} b_{ij}(q_i - q_j), \quad i \in F,
$$

and $\kappa_i$ is given in (11) in Section 4.1.

**Theorem 4.2.** Assume that $g^S$ satisfies Assumption 2.1 and $g^C$ satisfies Assumption 2.2. If $\mu_i > \|\triangle \theta_i\|_\infty$ and $\rho_i > \|\hat{q}_d\|_\infty, \forall i \in F$, then, by using the proposed distributed control law (13) with (11) and (14) for the followers’ dynamics (1), we can conclude that

1. $\hat{q}_i \to \hat{q}_d, \forall i \in F$.
2. $\hat{q}_d \to \text{co}(q_j, j \in L), \forall i \in F$.

**Proof.** Following the similar analysis in the proof of Theorem 4.1, we construct a Lyapunov function as:

$$
U = \frac{1}{2} \sum_{i=1}^{n} s_i^2 s_i + \frac{\alpha}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} + \alpha \sum_{i=1}^{n} \sum_{j=1}^{n+m} Q_{ij},
$$

It is easy to show that $U \leq 0$ when $0 < \alpha < 2\lambda_{\min} (R^T) \sqrt{\lambda}$, $\mu_i > \|\triangle \theta_i\|_\infty$ and $\rho_i > \|\hat{q}_d\|_\infty, \forall i \in F$. Therefore, the connectivity maintenance analysis follows from the proof of Theorem 3.1. Note that the connectivity maintenance result guarantees that $\lambda_{\min} (R^T) > 0$. Similar to the proof of Theorem 4.1, it follows that $s_i \to 0$ and $\hat{q}_i \to \hat{q}_d, \forall i \in F$, as $t \to \infty$. On the sliding surface, by Lemma 2.1, $\sum_{i=1}^{n+m} a_{ij}(q)(q_i - q_j) = 0$ implies $q_i \to \text{co}(q_j, j \in L), \forall i \in F$. This completes the proof. □

**Corollary 4.1.** If there is only one leader in the leader set, the convex hull formed by the leaders will reduce to a single point, i.e., with the proposed control law in this section, the final generalized coordinates of the followers will track that of the leader exactly.

**Remark 4.2.** A containment control algorithm for networked Lagrange systems is also proposed in [26] when the leaders’ generalized coordinate derivatives are time-varying. However, the model uncertainties are not considered and the design of the sliding mode estimator may increase the complexity of the algorithm in [26]. In contrast, the control law (13) of this paper is easier to implement and the control parameters only rely on the local information.

5. Simulation results

In this section, numerical simulation results are given to validate the effectiveness of the theoretical results obtained in this paper. We assume that there exist four followers ($n = 4$) and three leaders ($m = 3$) in the group. The system dynamics is given by [11,28]

$$
\begin{bmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_x \\
\dot{q}_y \\
\dot{q}_\theta
\end{bmatrix}
+ \begin{bmatrix}
-b_\theta \dot{q}_x - b_\theta \dot{q}_x & -b_\theta \dot{q}_x & 0 \\
-b_\theta \dot{q}_x & -b_\theta \dot{q}_x & 0
\end{bmatrix}
\begin{bmatrix}
\dot{q}_x \\
\dot{q}_y 
\end{bmatrix}
= \begin{bmatrix}
t_a \\
t_a \\
t_a
\end{bmatrix}, \quad i = 1, 2, 3, 4,
$$

where $M_{11} = a_1 + 2a_3 \cos q_y + 2a_4 \sin q_y, M_{12} = M_{21} = a_2 + a_3 \cos q_x + a_4 \sin q_y, M_{22} = a_2$, and $b = a_3 \sin q_y - a_4 \cos q_y$. In addition, we choose $a_1 = 8$, and $a_2 = a_3 = a_4 = 1$. 

1. Consider the case where only leader $L_1$ is present; the other leaders are not present.

2. Consider the case where the rest of the leaders are present.

3. Consider the case where the leaders’ generalized coordinate derivatives are time-varying.
For the case of the leaders’ generalized coordinate derivative being constant, the initial states of the followers are chosen as $q_1(0) = [-10, 10]^T$, $q_2(0) = [8, 10]^T$, $q_3(0) = [-10, -10]^T$, $q_4(0) = [0, -20]^T$, $\dot{q}_1(0) = [-0.1, 0.1]^T$, $\dot{q}_2(0) = [0.2, -0.2]^T$, $\dot{q}_3(0) = [0.7, -0.7]^T$, and $\dot{q}_4(0) = [0.4, -0.4]^T$. The trajectory of the leaders is chosen as $q_{L}(t) = [6, 0]^T$, $\dot{q}_{L}(t) = [0, -6]^T$, $\ddot{q}_{L}(t) = [0, 6]^T$, $\dot{q}_{L}(0) = [0, 6]^T$, $\dot{q}_{L}(0) = [0, 6]^T$, and $\ddot{q}_{L}(0) = [0, 6]^T$. The adjacency matrix $A_L$ of generalized coordinates associated with $g^L$ can be calculated by the initial states of the followers and the leaders. The initial setup for the leader generalized coordinate derivative estimators (6) is chosen as $\hat{v}_1(0) = [0.1, -0.1]^T$, $\hat{v}_2(0) = [-0.1, 0.1]^T$, $\hat{v}_3(0) = [0.3, -0.3]^T$ and $\hat{v}_4(0) = [-0.2, 0.2]^T$. The control parameters are chosen as $r = 20, d = 1, d_2 = 4$ and $\delta = 0.05$ for $i = 1, 2, 3, 4, k_i = 1, \forall i \in F$.

Fig. 1(a) shows the generalized coordinates of the leaders and the followers under control law (3) with (4), (5), (6) and (7) for the followers’ dynamics (1). It can be seen that the generalized coordinate derivatives of the followers converge to those of the leaders.

Fig. 2(a) shows the generalized coordinates of the leaders and the followers under control law (3) with (4), (5), (7) and (8) for the followers’ dynamics (1). It can be seen that the generalized coordinate derivatives of the followers are close to the convex hull formed by the leaders even if only the sensing information is available. Fig. 2(b) shows that the generalized coordinate derivatives of the followers converge to those of the leaders.

For the case of the leaders’ generalized coordinate derivative being time-varying, the initial states of followers are chosen as $q_1(0) = [-10, 5]^T$, $q_2(0) = [8, 7]^T$, $q_3(0) = [-10, -10]^T$, $q_4(0) = [5, -10]^T$, $\dot{q}_1(0) = [-0.1, 0.1]^T$, $\dot{q}_2(0) = [0.2, -0.2]^T$, $\dot{q}_3(0) = [0.7, -0.7]^T$, and $\dot{q}_4(0) = [0.4, -0.4]^T$. The initial states of the leaders are chosen as $q_{L}(0) = [6, 0]^T$, $\dot{q}_{L}(0) = [0, -6]^T$, $\ddot{q}_{L}(0) = [0, 6]^T$, $\dot{q}_{L}(0) = [0, 6]^T$, and $\ddot{q}_{L}(0) = [-1, 0]^T$. The adjacency matrix $A^L$ of generalized coordinates associated with $g^L$ can be calculated by the initial states of the followers and the leaders. The control parameters are chosen as $r = 20, d = 1, d_2 = 4, K = 1$, and $\delta = 1, \forall i \in F$.

Fig. 3(a) shows the generalized coordinates of the leaders and the followers under control law (9) with (10) and (11) for the followers’ dynamics (1). It can be seen that the generalized coordinate derivatives of the followers are close to the convex hull formed by the leaders when the leaders’ generalized coordinate derivatives are time-varying. Fig. 3 shows that the generalized coordinate derivatives of the followers converge to those of the leaders.

Fig. 4(a) shows the generalized coordinates of the leaders and the followers under control law (13) with (11) and (14) for the followers’ dynamics (1). It can be seen that the generalized coordinate derivatives of the followers converge into the convex hull formed by the leaders. Fig. 4(b) shows that the generalized coordinate derivatives of the followers converge to those of the leaders.

6. Conclusions

In this paper, the leader–follower swarm tracking control with group dispersion and cohesion behaviors was studied for a group of Lagrange systems. Both the cases of leaders’ generalized coordinate derivatives being constant and time-varying were considered. The proposed control algorithms were shown to achieve velocity matching, connectivity maintenance, collision avoidance and the followers were driven close to the cohesive inclusion formed by the leaders. In addition, the bound on the sum of the steady-state distances between the followers and the convex hull formed by the leaders was shown to be bounded and the bound was explicitly given. Numerical simulation verified these theoretical results. One interesting future research direction is the swarm tracking problem of multiple non-holonomic mobile agents.
(a) The generalized coordinates of the leaders and the followers. The circles denote the leaders and the big triangle is the convex hull spanned by the leaders. The squares and the crosses denote, respectively, the generalized coordinates of the followers at, respectively, \( t = 0 \) s, \( t = 200 \) s, and \( t = 300 \) s. The lines between the squares and crosses are the trajectories of the followers.

(b) The generalized coordinate derivatives of the followers and the leaders.

\[ \frac{\partial V_{ij}}{\partial q_i} = \begin{cases} (r^2 - \|q_i - q_j\|^2)^2(2\|q_i - q_j\|^2 + r^2 - 3d_1^2)(q_i - q_j), & d_1 < \|q_i - q_j\| \leq r, \\ 0, & \|q_i - q_j\| > r, \end{cases} \]

and

\[ \frac{\partial^2 V_{ij}}{\partial q_i^2} = \begin{cases} -2(r^2 - \|q_i - q_j\|^2)(2\|q_i - q_j\|^2 + r^2 - 3d_1^2), & \|q_i - q_j\| \leq r, \\ (-\|q_i - q_j\|^2 - 3\|q_i - q_j\|^2r^2 + 5\|q_i - q_j\|^2d_1^2 - d_1^2r^2) + 4(\|q_i - q_j\|^2 - d_1^2)(r^2 - \|q_i - q_j\|^2), & 0 < \|q_i - q_j\| < r, \\ \|q_i - q_j\|^2/((\|q_i - q_j\|^2 - d_1^2)^2r^4), & 0 < \|q_i - q_j\| > r. \end{cases} \]

For the case of \( d_1 < \|q_i - q_j\| < r \) when \( t = 0 \), \( V_{ij} \) is given by

\[ V_{ij} = \frac{1}{2(r^2 - \|q_i - q_j\|^2)} + \frac{d_2^2}{2 \|q_i - q_j\|^2 - d_1^2}, \]

for which,

\[ \frac{\partial V_{ij}}{\partial q_i} = \left( \frac{1}{(r^2 - \|q_i - q_j\|^2)^2} - \frac{d_2^2}{(\|q_i - q_j\|^2 - d_1^2)^2} \right) (q_i - q_j). \]
and

\[ \frac{d^2 V_{ij}}{dt^2} = \frac{r^2 + 3 \|q_i - q_j\|^2}{(r^2 - \|q_i - q_j\|^2)^{3/2}} + d_2^2 \left( \frac{d_2^2 + 3 \|q_i - q_j\|^2}{(\|q_i - q_j\|^2 - d_2^2)^{3/2}} \right). \]

We also assume that \( \|q_i - q_j\| > d_1 \) when \( t = 0 \). It will be shown in Appendix A.3 that \( \|q_i(t) - q_j(t)\| > d_1 \) for \( t \geq 0, \forall i, j \in \mathcal{V}, i \neq j \), if \( \|q_i(0) - q_j(0)\| > d_1 \) by using the proposed control laws. Note that \( V_{ij} \) achieves its local minimum when \( \|q_i - q_j\| = \sqrt{d_2^2 + d_1^2} \), where \( d_2 \) is used to adjust the minimum value of \( V_{ij} \). Also note that

\[ d \left( \sum_{j=1}^{n+m} \frac{\partial^2 V_{ij}}{\partial q_i^T} \right) = \sum_{j=1}^{n+m} (q_i - q_j)^2 \frac{\partial^2 V_{ij}}{\partial q_i^T}. \]

A.2. The adjacency matrix for the sensing graph

In order to design a smooth control law, we give a proper definition for the adjacency matrix \( A^S = [a_{ij}(q)] \) associated with \( q^S \), where \( q = [q_1^T, q_2^T, \ldots, q_{n+m}^T]^T \in \mathbb{R}^{n+m} \). Let

\[ Q_{ij} = \begin{cases} \frac{(r^2 - \|q_i - q_j\|^2)^3}{6r^4}, & 0 < \|q_i - q_j\| \leq r, \\ 0, & \|q_i - q_j\| > r. \end{cases} \]

\( a_{ij} \) is defined as

\[ a_{ij}(q) = \begin{cases} \frac{(r^2 - \|q_i - q_j\|^2)^2}{r^4}, & 0 < \|q_i - q_j\| \leq r, \\ 0, & \|q_i - q_j\| > r. \end{cases} \]

Note that \( \frac{\partial a_{ij}}{\partial q} = a_{ij}(q)(q_i - q_j) \). Each element \( a_{ij}(q) \) of \( A \) is nonnegative, differentiable and a function of \( \|q_i - q_j\| \). Also note that the boundedness of \( Q_{ij} \) guarantees the boundedness \( a_{ij}(q) \) and \( \frac{\partial a_{ij}}{\partial q} \).

A.3. Proof of Theorem 3.1

Proof. 1) Connectivity maintenance analysis

Motivated by [29], [30], and [31], we construct a Lyapunov function candidate as

\[ U = \frac{1}{2} \sum_{j=1}^{n+m} s_j^T M s_j + \frac{\delta}{2} \sum_{j=1}^{n+m} V_{ij} + \frac{n}{2} \sum_{j=1}^{n+m} \sum_{j=1}^{n+m} V_{ij} + \frac{n}{2} \sum_{j=1}^{n+m} Q_{ij} \]

\[ + \frac{n}{2} \sum_{j=1}^{n+m} \sum_{j=1}^{n+m} Q_{ij} + \frac{1}{2} \sum_{j=1}^{n+m} \|\Delta \theta_i\|^2, \]

where \( \Delta \theta_i \) is defined in Section 2.1. Taking the derivative of \( U \), we have

\[ \dot{U} = \sum_{i=1}^{n+m} \left( Y_i \Delta \theta_i - k_s s_i - \delta \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) - \delta \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \right) \]

\[ - \sum_{i=1}^{n+m} \Delta \theta_i^T Y_i^T s_i + \delta \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) + \sum_{i=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \]

\[ + \delta \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \left( \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) + \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \right) \]

\[ + \sum_{i=1}^{n+m} \left( \Delta \theta_i - \dot{\Delta} \theta_i - \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) \right) \]

\[ - \sum_{i=1}^{n+m} \left( \Delta \theta_i^T \dot{Y}_i + \delta \sum_{i=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \right) \]

\[ - \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) - \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \]

\[ + \delta \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \left( \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) + \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \right) \]

\[ + \sum_{i=1}^{n+m} \left( \Delta \theta_i - \dot{\Delta} \theta_i - \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) \right) \]

\[ - \sum_{i=1}^{n+m} \left( \Delta \theta_i^T \dot{Y}_i + \delta \sum_{i=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \right) \]

\[ - \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) - \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \]

\[ - \sum_{i=1}^{n+m} \left( \Delta \theta_i - \dot{\Delta} \theta_i - \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) \right) \]

\[ - \sum_{i=1}^{n+m} \left( \Delta \theta_i^T \dot{Y}_i + \delta \sum_{i=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \right) \]

\[ - \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} a_{ij}(q)(q_i - q_j) - \sum_{j=1}^{n+m} \frac{\partial V_{ij}}{\partial q_i} \]
where we have used the facts that \( \frac{\partial U}{\partial q_i} = \frac{\partial U}{\partial q_j} \), \( \forall i \in F, j \in L \), 
\[ \dot{q}_d \sum_{i=1}^{n} \sum_{j=1}^{m} \left( a_i(q_i, q_j) + \frac{\partial V_q}{\partial q_i} \right) = 0 \] 
and \( \frac{\partial V_q}{\partial q_i} = -\frac{\partial V_q}{\partial q_j}, \forall i, j \in F. 
\]

The fact that \( g^C \) satisfies Assumption 2.2 implies that \( \lambda_{\min}(T^C) > 0 \), where \( T^C \) is as defined in Section 2.3 associated with the graph \( g^C \). Therefore, we have

\[ \dot{U} \leq -\sum_{i=1}^{n} k_i s_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n-m} a_i(q_i, q_j) \left( \frac{\partial V_q}{\partial q_i} - \frac{\partial V_q}{\partial q_j} \right)^2 + \sum_{i=1}^{n} \sum_{j=1}^{n-m} a_i(q_i, q_j) \left( \frac{\partial V_q}{\partial q_i} - \frac{\partial V_q}{\partial q_j} \right)^2 - \lambda_{\min}(T^C) \sum_{i=1}^{n} \| \dot{q}_i - \dot{q}_d \|^2 \leq 0. \]

This implies that \( U(t) \) is bounded for \( t \geq 0 \) and hence \( \| q_i - q_j \| \) is bounded for all \( i, j \in \mathcal{V} \) and \( t \geq 0 \). On the other hand, the definition of \( V_q \) implies that \( \lim_{\| q_i - q_j \| \to 0} V_q = \infty \). Thus, we know that no edge will be lost at switching times, which implies that \( N_0^q(0) \leq N_0^q(t) \) for all \( i \in F \) and \( t \geq 0 \).

2) Velocity matching analysis

From the fact that \( U(t) \) is bounded, we know that \( s_i, \triangle \dot{q}_i, \dot{v}_i, \dot{q}_d \) and \( \dot{q}_q \) are bounded. Since the boundedness of \( V_q \) and \( \dot{q}_q \) guarantee the boundedness of \( \frac{\partial V_q}{\partial q_i} \) and \( \frac{\partial V_q}{\partial q_j} \), we know that \( \sum_{i=1}^{n} \frac{\partial V_q}{\partial q_i} = 0 \) and further that \( \dot{q}_i - \dot{q}_d \) is bounded in view of (7). In view of (8), it follows that \( \hat{v}_i \) is bounded from the fact that \( \dot{q}_q \) is bounded. Since the boundedness of \( V_q \) and \( \dot{q}_q \) also guarantees the boundedness of \( \frac{\partial V_q}{\partial q_i} \) and \( \frac{\partial V_q}{\partial q_j} \), it is easy to show that \( \frac{d}{dt} \left( \sum_{i=1}^{n} \frac{\partial V_q}{\partial q_i} + \sum_{i=1}^{n-m} a_i(q_i, q_j) \right) \) is bounded. Thus, we know that \( \dot{q}_q \) is bounded and \( \hat{v}_i \) is bounded. Then, from the closed-loop dynamics

\[ M_i(q_i) s_i + C_i(q_i, \dot{q}_i) \dot{s}_i = Y(q_i, \dot{q}_i, \dot{q}_q) \Delta \dot{q}_i - k_i s_i, \quad i \in F, \]

we know that \( \dot{s}_i \) is bounded. This implies that \( \dot{U} \) is bounded. Then, by the Barbalat’s lemma, we have \( U \to 0 \) as \( t \to \infty \). Therefore, we know that \( \dot{s}_i \to 0, \dot{q}_q \to \dot{q}_d, \forall i \in F \) as \( t \to \infty \). This shows that the velocity matching is achieved for each follower.

3) Group dispersion analysis

Because \( U(t) \) is bounded, it is easy to show that \( \| q_i - q_j \| \) is bounded for all \( i, j \in \mathcal{V} \) and \( t \geq 0 \). We also know that \( \lim_{\| q_i - q_j \| \to 0} V_q = \infty \). Therefore, it follows that \( \| q_i(t) - q_j(t) \| \to d_1 \) for all \( i, j \in \mathcal{V} \) but \( i \neq j \).

4) Containment boundedness analysis

Since \( s_i \to 0 \) and \( \dot{q}_q \to \dot{q}_d \), \( \forall i \in F \), as \( t \to \infty \), we know that \( \sum_{i=1}^{n} a_i(q_i, q_j) \) and \( \sum_{i=1}^{n-m} a_i(q_i, q_j) = 0 \) as \( t \to \infty \). Thus, we have that \( \dot{q}_q = \inf \| \dot{q}_i - \dot{q}_j \|, \forall i \in \partial \mathcal{O} \), defined in Definition 3.1 is bounded by

\[ \dot{q}_q \leq \frac{1}{\lambda_{\min}(T^C)} \sum_{i=1}^{n} \sum_{j=1}^{m} a_i(q_i, q_j) \leq \frac{1}{\lambda_{\min}(T^C)} \sum_{i=1}^{n} \sum_{j=1}^{m} a_i(q_i, q_j). \]

We now have used the fact that \( \frac{1}{\lambda_{\min}(T^C)} \sum_{i=1}^{n} \sum_{j=1}^{m} a_i(q_i, q_j) \) is in the convex hull formed by the leaders because \( T^{-1} \mathcal{D} \otimes I \mathcal{D} \), \( q_i \in \mathcal{B}_i, \dot{q}_j \in \mathcal{B}_j, \dot{q}_q \in \mathcal{B}_q, \dot{q}_d \in \mathcal{B}_d, \).

(see Section 2.3) Note that the connectivity maintenance result guarantees that \( \lambda_{\min}(T^S) > 0 \).

Consider any \( i \in F \) and \( j \in \mathcal{V} \). If \( \| q_i(0) - q_j(0) \| > r \), we know that \( \| V_q(0) \| = 0 \) and \( \frac{\partial V_q}{\partial q_i} = 0 \). Let \( \eta_{i(j)} > 0 \) be such that

\[ \frac{r^2 - \eta_{i(j)}^2}{\eta_{i(j)}^2} = U(0). \]

Then, based on the fact that \( U(t) \leq U(0) \) and monotonically of the function \( \frac{1}{\eta_{i(j)}^2} \), we have that \( \| q_i - q_j \| \geq \eta_{i} = \min_{j} \eta_{i(j)}. \) Note that \( \lambda_{\min}(T^C) \) is bounded. Therefore, for the case of \( \| q_i(0) - q_j(0) \| > r \), we have that

\[ \| \frac{\partial V_q}{\partial q_i} \| \leq \frac{6(r^2 - r^2)}{U(t) - d_1^2} \frac{1}{r^2}. \]

Similarly, if \( d_1 < \| q_i(0) - q_j(0) \| \leq r \), let \( \eta_{i(j)} > 0 \) be such that \( \frac{r^2 - \eta_{i(j)}^2}{\eta_{i(j)}^2} = U(0). \) Then, based on the fact that \( U(t) \leq U(0) \) and monotonically of the function \( \frac{1}{\eta_{i(j)}^2} \), we have that \( \| q_i - q_j \| \geq \eta_{i} = \min_{j} \eta_{i(j)}. \) Further, from the fact of \( \| q_i(0) - q_j(0) \| \leq r \), we have that

\[ \| \frac{\partial V_q}{\partial q_i} \| \leq \frac{1}{U(t) - d_1^2} \frac{1}{r^2}. \]

In all cases, we have that

\[ \| \frac{\partial V_q}{\partial q_i} \| \leq \alpha i \leq \max \left\{ \frac{6(r - r^2)}{U(t) - d_1^2}, \frac{1}{U(t) - d_1^2} \right\}, \]

and \( \varphi \leq \min \left\{ \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} n \alpha \| q_i - q_j \| \| V_q \| \| \frac{\partial V_q}{\partial q_i} \| \right\} \).
\[
- \sum_{j=1}^{n} (\rho_j - \|\dot{q}_j\|_\infty) \sum_{i=1}^{n} T^C_{ij} s_j \leq -K\lambda_{\min}(T^C) \sum_{i=1}^{n} s_i^2
\]

\[
+ \alpha \sum_{i=1}^{n} (\dot{q}_i - \dot{q}_d)^T \left( s_i - \sum_{j=1}^{n+m} b_{ij}(\hat{q}_i - \hat{q}_j) \right) \leq -K\lambda_{\min}(T^C) \sum_{i=1}^{n} s_i^2 + \alpha \sum_{i=1}^{n} (\dot{q}_i - \dot{q}_d)^T s_i
\]

where \( T^C_{ij} \) is the \((i, j)\)th entry of matrix \( T^C \) (defined in Section 2.3) associated with \( g^C \), and we have used the facts that \( \mu_i > \|\Delta \theta_i\|_\infty \), \( \rho_i > \|\dot{q}_i\|_\infty, \) \( \forall i \in F, \forall j \in L, \) \( \dot{q}_d \sum_{j=1}^{n+m} a_{ij}(q)(\dot{q}_i - \dot{q}_j) + \sum_{j=1}^{n+m} \frac{\partial V_j}{\partial q_j} = 0 \) and \( \frac{\partial V_j}{\partial q_j} = -\frac{\partial V_j}{\partial q_j}, \) \( \forall i, j \in F. \) Then, if \( \alpha \) is selected as \( 0 < \alpha < 2\lambda_{\min}(T^C) \sqrt{K}, \) we have that \( \dot{U} \leq 0. \) Therefore, the connectivity maintenance analysis follows from Theorem 3.1.

Similarly to the analysis given in Appendix A.3, we know that \( s_i, \Delta \theta_i, \) and \( \frac{d}{dt} \left( \sum_{j=1}^{n+m} a_{ij}(q)(\dot{q}_i - \dot{q}_j) + \sum_{j=1}^{n+m} \frac{\partial V_j}{\partial q_j} \right) \) are bounded. It follows from the closed-loop dynamics (12) that \( \dot{q}_d \) is bounded. This implies that \( \dot{U} \) is bounded when \( \sum_{j=1}^{n+m} a_{ij}(q)(\dot{q}_i - \dot{q}_j) + \sum_{j=1}^{n+m} \frac{\partial V_j}{\partial q_j} = 0, \forall i \in F. \) It is easy to show that \( \dot{U} \) is also bounded when \( \sum_{j=1}^{n+m} a_{ij}(q)(\dot{q}_i - \dot{q}_j) = 0, \forall i \in F. \) Then, by the Barbala's lemma, we have \( \dot{U} \to 0 \) as \( t \to \infty. \) Therefore, it follows that \( s_i \to 0 \) and \( \dot{q}_i \to \dot{q}_d, \forall i \in F, \) as \( t \to \infty. \) Then, the velocity matching analysis, group dispersion analysis and containment boundedness analysis all follow from the proof of Theorem 3.1.

References


