Continuous weak measurement of quantum coherent oscillations

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We consider the problem of continuous quantum measurement of coherent oscillations between two quantum states of an individual two-state system. It is shown that the interplay between the information acquisition and the backaction dephasing of the oscillations by the detector imposes a fundamental limit, equal to four, on the signal-to-noise ratio of the measurement. The limit is universal, e.g., independent of the coupling strength between the detector and system, and results from the tendency of quantum measurement to localize the system in one of the measured eigenstates.

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Coherent oscillations between the two states of a quantum two-state system represent one of the most basic and direct manifestations of quantum mechanics and are encountered in practically all areas of physics. The question of how to measure them directly in an individual two-state system was apparently formulated\(^1-4\) for the first time in the context of quantum dynamics of Josephson junctions, where the oscillating variable, the magnetic flux in a superconducting loop, is macroscopic. A common feature of the measurement schemes suggested in this context is the use of conventional “projective” measurements that localize the flux in one of its eigenstates and suppress the oscillations. The time evolution of the oscillations can then only be studied if the experiment is repeated many times with the same initial conditions, and the information about oscillations is contained in the probability distribution of the measurement outcomes. This means that the oscillations are effectively studied in an ensemble of systems, not in an individual system. Another, practical, disadvantage of the projective measurements is the need to control the system dynamics externally on a time scale shorter than the oscillation period, in order to allow for preparation of the initial state of the system, free evolution of the oscillations, and subsequent measurement. Since the oscillation frequency is limited from below by several factors, including decoherence time and temperature, this requirement presents at the very least a challenging technical problem. Although this problem has been solved for oscillations of charge in a Cooper-pair box,\(^5\) it presents considerable obstacle to direct time-domain observation of quantum-coherent oscillations in other systems, e.g., oscillations of magnetic flux in superconducting quantum interference devices (SQUID’s), which has been observed only through the spectroscopy of energy levels.\(^6\)

The aim of our work is to point out that the problem of measurement of quantum-coherent oscillations in an individual two-state system can be somewhat simplified if projective measurements are replaced with a weak continuous measurement, and to study the quantitative characteristics of such a measurement scheme. As is emphasized frequently in the theory of quantum measurements (see, e.g., Refs. 7–9), the “textbook” projective quantum measurement requires the dynamic interaction between the system and the detector to be sufficiently strong to establish nearly perfect correlation between their states. If the interaction is weak, this does not happen, and the measurement provides only limited information about the system. Such a weak measurement, however, perturbs the system only slightly and can be performed continuously. Below we consider quantitatively the process of continuous weak measurement of quantum-coherent oscillations. We calculate the spectral density of the detector output and show that the trade off between the acquisition of information and dephasing due to the detector backaction on the oscillations imposes a fundamental limit, equal to four, on the signal-to-noise ratio of the measurement. In this work, we use a more conventional nonelective approach to measurement, which discusses only the quantities averaged over the detector output. All the results can be reproduced within the selective description of the measurement process.\(^10\)

Although the main conclusions of our work are quite general, in what follows we prefer to use the language of a particular system: two coupled quantum dots measured with a quantum-point contact. Quantum-point contacts were used as electron detectors in Refs. 11–13 and described theoretically in Refs. 14–18,10. Coherent-electron oscillations in coupled dots were observed indirectly in the dc transport under microwave irradiation.\(^19\)

The Hamiltonian of the system [see inset in Fig. 1(a)] is

\[
H = -\frac{1}{2}(\epsilon \sigma_z + \Delta \sigma_z + \sigma_z U) + \sum_{ik} \epsilon_i \sigma_k a_k^\dagger a_{ik},
\]

\[
U = \sum_{ij} U_{ij} \sum_{kp} a_{ik}^\dagger a_{jp}.
\]

The first two terms here describes an electron oscillating between the two discrete energy states localized in the quantum dots: \(\epsilon\) is the energy difference between the states, \(-\Delta/2\) is their tunnel coupling, and the \(\sigma\’s\) denote Pauli matrices. The operators \(a_{ik}\) represent point-contact electrons in the two scattering states \(i = 1, 2\) (incident from the two contact electrodes) with momentum \(k\). The coupling \(\sigma_z U/2\) is due to an additional scattering potential \(\pm U(\chi)/2\) created in the point contact by the electron occupying one or the other dot. The point contact is biased with a dc voltage \(V\), so that changes in the scattering potential lead to changes in the current \(I\) through the contact. We take \(eV\) to be much smaller than both the Fermi energy in the point contact and the inverse
traversal time of the contact. This allows us to linearize the
energy spectrum of the point-contact electrons: \( \varepsilon \equiv v_F k \),
where \( v_F \) is the Fermi velocity, and neglect the momentum
dependence of the coupling matrix elements \( U_{ij} = \int dx \psi_i^\dagger(x) U(x) \psi_j(x) \) of the perturbation \( U \) in the basis of
the two scattering states \( \psi_i(x) \). We also assume that the \( U_{ij} \)
are sufficiently small for the point contact to operate as a
linear detector, and treat the contact’s response to electron in
the dots in the linear-response approximation.

Quantum oscillations of electron between the dots create
an oscillating component of the current \( I \) through the point
contact. Since the phase of the oscillation diffuses under the
backaction of the shot noise of the point contact, the oscilla-
tions are best characterized by their spectral density. To find
the spectral density of the current \( I \), we choose the origin of
the coordinate \( x \) along the contact in such a way that the unperturbed
scattering potential is effectively symmetric, i.e., the reflection amplitudes for both scattering states are the
same. Then, the current operator calculated at a point \( x \) in the
asymptotic region of the scattering states is

\[
I = \frac{e v_F}{L} \sum_p \left[ D(a_{1p}^\dagger a_{1p} - a_{2p}^\dagger a_{2p})
+ i(DR)^{1/2} e^{-i(k-p)|x|} (a_{1p}^\dagger a_{2p}^\dagger - a_{2p}^\dagger a_{1p}^\dagger) \right],
\]

(2)

where \( D \) and \( R = 1 - D \) are the transmission and reflection
probabilities of the point contact, \( L \) is a normalization length,
and the variation of the momentum near the Fermi points
(i.e., the difference between \( k \) and \( p \)) was neglected every-
where besides the phase factor in the second term. The
reason for keeping this factor will become clear later.

In the linear-response regime, the current response of the
point contact is driven by the part of the perturbation \( U \)
causing transitions between the two scattering states \( \psi_{12} \).
Considering the effect of this perturbation on the stationary
(symmetric and antisymmetric) combinations of the scattering
states, one can show that the real part of the transition
matrix element \( U_{12} \) is related to the change \( \delta D \) of the trans-
mision probability of the contact:

\[
U_{12}^{\Re} = \frac{v_F}{L} \frac{\delta D + i u}{2(DR)^{1/2}} , \quad U_{21}^{\Re} = U_{12}^{\Re} .
\]

(3)

The imaginary part of \( U_{12} \), expressed through a dimen-
sionless parameter \( u \) in Eq. (3), does not affect the current \( I \). Qualitatively, it characterizes the degree of asymmetry in the
coupling of the quantum dots to the point contact; \( u \neq 0 \) if the
perturbation potential \( U(x) \) is applied symmetrically with respect to the main scattering potential of the point contact.

When the point contact is used as a detector in a quantum
measurement, the current \( I \) plays the role of the measurement
output and should behave classically. This condition requires
the spectral density of \( I \) to be much larger than the spectral
density of the zero-point fluctuations in the relevant fre-
quency range. It is satisfied when the voltage \( V \) across the
point contact, which determines the magnitude and the threshold frequency of the shot noise of \( I \), is sufficiently
large, \( eV \gg \varepsilon, \Delta \). For the point contact to be an effective
detector, \( eV \) should also be much larger than the temperature \( T \).
In this regime, it is straightforward to find the correlation
functions of the perturbation \( U \) and the current \( I \) at frequen-
cies much smaller than \( eV \), in the zeroth order in \( U \) from
Eqs. (1), (2), and (3):

\[
\langle U(t) U(t + \tau) \rangle_0 = \frac{e^2 V}{2\pi} \frac{(i \delta D + u) \delta(\tau - \eta)}{\delta(\tau - \eta)}. \quad (4)
\]

(5)

The spectral density of \( I \) at low frequencies is dominated by the regular shot noise, and the current-correlation function is
\( K_1^{(0)}(\tau) \equiv \langle I(i) I(t + \tau) \rangle_0 - \langle I \rangle^2 = e^2 V D/I(\tau) \), where \( \langle I \rangle \equiv e^2 VD/I(\pi) \). The time delay \( \eta = |x|/v_F \) in Eq. (4) comes from the
phase factor \( e^{-i(k-p)|x|} \) kept in Eq. (2), and is infinitesim-
ally small for small traversal time of the contact. It is nev-
ertheless important for resolving the ambiguity in averages
involving the time ordering of \( I \) and \( U \) that are needed for the
calculation of the current response: \( i\hbar d/dt \langle \mathcal{T}[I(i) U(t')] \rangle_0 \).
Equation as a detector. In this regime, the point contact represents symmetric coupling corresponds to an optimum in its operation. It does not affect the current response of the point contact, and averaged over its dynamics. The current change established as a result of the interaction with the point contact and averaged over its dynamics. The current change \( \delta I \) is the time-evolution operator. Taking the trace over the electron states in the point contact in Eq. (6) with the help of the correlation functions (4) and (5), we get

\[
K_f(\tau) = K_f^0(\tau) + \frac{(\delta I)^2}{4} \langle \sigma_z \sigma_z(\tau) \rangle.
\]

The average \( \langle \cdots \rangle \) in Eq. (7) is taken over the two states of the quantum dots with the stationary dot density matrix \( \rho \) established as a result of the interaction with the point contact and averaged over its dynamics. The current change \( \delta I = e^2(\delta D)V/\pi \) is the current response to electron oscillations between the dots, and \( \sigma_z(\tau) \) now denotes the full time evolution of \( \sigma_z \), driven both by the dot Hamiltonian and the interaction \( U \) with the point contact. Qualitatively, Eq. (7) shows that the current correlation function directly reflects the correlation function of the electron position in the dots given by \( \sigma_z \).

The time dependence of the operator \( \sigma_z(\tau) \) in Eq. (7) is obtained by tracing out the point-contact degrees of freedom in Eq. (6) with the help of the \( U-U \) correlation function (4). In this way we get the standard set of equations for the matrix elements \( \sigma_{ij} \) of \( \sigma_z(\tau) \):

\[
\dot{\sigma}_{11} = \Delta i \sigma_{12}, \quad \dot{\sigma}_{12} = (i\epsilon - \Gamma)\sigma_{12} - i\Delta \sigma_{11},
\]

and \( \sigma_{22} = -\sigma_{11} \). The rate

\[
\Gamma = eV(\delta D)^2 + u^2 / 8\pi DR
\]

describes backaction dephasing of the coherent-electron oscillations between the dots by the point contact. Equation (9) shows that the dephasing rate reaches a maximum in the case of symmetric dot-contact coupling \( (u=0) \). In this case, the rate of dephasing by a point contact has been found in Refs. 14–16 for a single quantum-dot. In the double-dot case the situation is quite different in that the dephasing rate \( \Gamma \) manifests itself directly as the width of the spectral line of the quantum-coherent oscillations. Increased dephasing in the case of asymmetric dot-contact coupling was discussed qualitatively in Ref. 18 and studied experimentally in Ref. 13. Since the decrease of \( \Gamma \) with increasing asymmetry \( u \) does not affect the current response of the point contact, symmetric coupling corresponds to an optimum in its operation as a detector. In this regime, the point contact represents an ideal quantum detector in a sense that the minimum value

of the dephasing rate \( \Gamma \) is determined purely by the rate of information acquisition about the state of the quantum dots and can be written as \( \Gamma = (\delta I)^2/4S_0 \), where \( S_0 = 2e(I)R \) is the spectral density of the current shot noise of the point contact. This part of the dephasing is fundamentally unavoidable and reflects the tendency of quantum measurement to localize the measured system in one of the eigenstates of the measured observable, in our case, the electron position \( \sigma_z \).

The dot density matrix \( \rho \) satisfies the same set of equations (8), except for the normalization, \( \rho_{11} + \rho_{22} = 1 \), and its stationary value is \( \rho = 1/2 \). Solving Eqs. (8) with the initial condition \( \sigma_z(0) = \sigma_z \) and averaging \( \sigma_z(\tau) \) over \( \rho = 1/2 \) we find the correlation function (7) and the spectral density \( S_f(\omega) = \int_{-\infty}^\infty \Gamma f(\tau) e^{i\omega \tau} d\tau \) for \( \epsilon = 0 \):

\[
S_f(\omega) = S_0 + \frac{\Gamma \Omega^2(\delta I)^2}{(\omega^2 - \Omega^2)^2 + \Gamma^2 \omega^2}.
\]

In the case of biased dots with \( \epsilon \neq 0 \), it is convenient to calculate the spectrum numerically from Eq. (8). The spectrum in this case is plotted in Fig. 1 for several values of \( \epsilon \) and the dephasing rate \( \Gamma \). For weak dephasing, \( \Gamma \ll \Delta \), the spectrum consists of a zero-frequency Lorentzian that vanishes at \( \epsilon = 0 \) and grows with increasing \( |\epsilon| \), and a peak at the oscillation frequency \( \Omega = (\Delta^2 + \epsilon^2)^{1/2} \). Although the width of the oscillation peak is \( \Gamma \) and can be small for sufficiently weak dot-contact coupling, its height cannot be arbitrarily large in comparison to the background-noise spectral density \( S_0 \). At \( \epsilon = 0 \), when the amplitude of the oscillations is maximum, the peak height is \( S_{\text{max}} = (\delta I)^2/\Gamma \). Even in this case, the ratio of the peak height to the background is limited:

\[
\frac{S_{\text{max}}}{S_0} = \frac{4(\delta D)^2}{(\delta D)^2 + u^2} \leq 4.
\]

This limitation is universal, e.g., independent of the coupling strength between the dots and the point contact, and reflects quantitatively the interplay between measurement of the quantum coherent oscillations and their backaction dephasing. The fact that the height of the spectral line of the oscillations can not be much larger than the noise background means that, in the time domain, the oscillations are drowned in the shot noise.

The total intensity of the oscillation line in the spectrum:

\[
\int_0^\infty (S_f(\omega) - S_0) d\omega = \frac{(\delta I)^2}{4}
\]

does depend on the strength of coupling to the point contact, increasing as the coupling becomes stronger. An interesting feature of Eq. (12) is that it stresses the impossibility of a simple classical interpretation of the quantum-coherent oscillations, since the intensity of harmonic classical oscillations of the same amplitude \( \delta I/2 \) would be two times smaller, and no classical signal of this amplitude could produce the oscillation line with intensity (12).
When the backaction dephasing rate $\Gamma$ increases, the oscillation line broadens towards the lower frequencies, and eventually turns into the growing spectral peak at zero frequency that reflects the incoherent electron jumps between the two dots. At large $\Gamma$, when the coherent oscillations are suppressed, the rate of incoherent tunneling decreases with increasing $\Gamma$. For instance, at $\Gamma=\Omega$, the tunneling rate is $\gamma = \Delta^2/2\Gamma$ and the spectral density of the point-contact response has the standard Lorentzian form $S(k) = 2\gamma(\delta k)^2/(4\gamma^2 + \omega^2)$. Suppression of the tunneling rate $\gamma$ with increasing dephasing rate $\Gamma$ is an example of the generic “quantum zeno effect” in which quantum-measurement suppresses the decay rate of a metastable state. In the context of search for the macroscopic quantum coherent oscillations, the Lorentzian spectral density has been observed and used for measuring the tunneling rate of incoherent quantum flux tunneling in SQUID's.21

The maximum signal-to-noise ratio $S_{max}/S_0$ (11) is attained if the fundamental backaction of the detector is the only dephasing mechanism of the coherent oscillations. In the case of measurement with a point contact, the fundamental measurement-induced dephasing caused above is created by the backscattering part $U_{12}$ (1) of the dot-contact interaction that dominates at large bias voltages $V$. The forward scattering $U_{11}, U_{22}$ does not affect the current $I$ in the contact but creates a weak additional dephasing and energy-relaxation mechanism for the oscillations. We now want to discuss the effect of such a weak relaxation on the spectral density of the oscillations noticeable if the backaction dephasing is also weak, $\Gamma \ll \Delta$

The inclusion of the additional weak relaxation does not modify the calculations that lead to Eq. (7), apart from a trivial modification that now the average $\sigma_z$ is nonvanishing, and the current-correlation function should be calculated as $K_i(\tau) = K_i(\tau) + (\delta I/2)^2[(1/2)(\sigma_x \sigma_x)(\tau) + (\sigma_y \sigma_y)(\tau)] - \langle \sigma_y \sigma_y \rangle^2$. For weak coupling, it is convenient to find the time evolution of $\sigma_y(\tau)$ in the basis of eigenstates of the two-state Hamiltonian $\{ (\epsilon \sigma_z + \Delta \sigma_x)/2 \}$. Solving the Heisenberg equation of motion up to the second order in the dot-contact coupling, and tracing out the contact degrees of freedom, we get a set of equations for the evolution of the matrix elements $s_{ij}$ of $\sigma_z(\tau)$ in the eigenstates basis:

\[
\dot{s}_{ij}(\tau) = \Gamma_{e} \left[ \frac{\epsilon}{\Omega} - \text{coth} \left( \frac{\Omega}{2T} \right) s_{jj} \right] + (-1)^s \frac{\Delta^2}{2\Omega^2} (s_{11} - s_{22}),
\]

\[
\dot{s}_{12}(\tau) = (i\epsilon - \Gamma_{0})s_{12},
\]

with the initial conditions $s_{11} = -s_{22} = \epsilon/\Omega$, and $s_{12} = -\Delta/\Omega$. The characteristic energy-relaxation rate in Eq. (13) is $\Gamma_{e} = \nu \Delta^2/\Omega$, where $\nu = (1/\pi)(U_{11}' + U_{22}')/(LV_F)^2$, and the total dephasing rate is

\[
\Gamma_0 = \nu \Delta^2/\Omega + 4\epsilon^2/\nu + 2\Delta^2/\Omega^2.
\]

The dot density matrix $\rho$ in the eigenstates basis satisfies similar equations, and the stationary values of its matrix elements are $r_{11} = (\Gamma_{e} + \Gamma_{0})/2\Gamma_{e}$ and $r_{12} = 0$, where $\Gamma_{e} = \Gamma_{e} \text{coth}(\Omega/2\nu) + \Delta^2/\Omega^2$. From these relations and Eqs. (13) we find the spectral density:

\[
S(i) = S_0 + \frac{(\delta I)^2}{\Omega^2} \left\{ \epsilon^2 \left[ 1 - \frac{\Gamma_{e}}{\Gamma_{0}} \right]^2 \frac{\Gamma_{e}}{\omega^2 + \Gamma_{0}} \right. + \left. \frac{\Delta^2}{2} \sum \frac{\Gamma_{0}}{\omega^2 + \Gamma_{0}^2} \right\}.
\]

As before, the spectral density consists of a zero-frequency Lorentzian and peaks at $\pm \Omega$ of width $\Gamma_0$ that represent the coherent electron oscillations. Energy relaxation with characteristic rate $\Gamma_{e}$ broadens the oscillation peak and reduces its height $S_{max}$, so that the relative magnitude of the peak, $S_{max}/S_0$, decreases in comparison with its value without relaxation.

In summary, we have considered a continuous weak quantum measurement by a point contact of quantum-coherent oscillations in a two-state system, and calculated the spectral density of the output signal of the measurement. It has been shown that the backaction dephasing introduced into the oscillation dynamics by the measurement imposes the fundamental limit on its signal-to-noise ratio. We also calculated the effect of energy relaxation on the output spectrum.

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