Supplemental Material for “Multi-time correlators in continuous measurement of qubit observables”

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Section A: Proof of the “collapse recipe”

In this section we prove the “collapse recipe”, which states that in the absence of phase backaction, the multi-time correlator

\[ K_{t_1,t_2,...,t_N}(t_1,t_2,...,t_N) \equiv \langle I_{t_N}(t_N)\ldots I_{t_2}(t_2) I_{t_1}(t_1) \rangle \]  

(1)
can be calculated by replacing the actual continuous measurement with projective measurement of operators \( \sigma_{t_k} \) at time moments \( t_k \) \( (k = 1, 2, ... , N, t_1 < t_2 < ... < t_N) \), while the qubit evolution at \( t \neq t_k \) is replaced with the ensemble-averaged evolution. In the proof we show that the value of the correlator \( K \) obtained in this way coincides with the value obtained from the quantum Bayesian formalism, in which the qubit evolution is described by the stochastic equation (in Itô interpretation)

\[ \dot{r} = \Lambda_{\text{ens}}(r - r_{st}) + \sum_{\ell=1}^{N_s} n_{\ell} \epsilon_{\ell}(t) \frac{r}{\sqrt{\tau_{\ell}}} + \frac{n_{\ell} r}{\sqrt{\tau_{\ell}}} \epsilon_{\ell}(t), \]  

(2)

where \( r \) is the vector of Bloch-sphere components, \( \rho = \frac{1}{2} + r \sigma / 2 \), the output signal of the \( \ell \)th detector continuously measuring the qubit operator \( \sigma_{\ell} = n_{\ell} \sigma \) is

\[ I_{\ell}(t) = n_{\ell} r(t) + \sqrt{\tau_{\ell}} \epsilon_{\ell}(t), \]  

(3)

\( \tau_{\ell} \) is the corresponding measurement (collapse) time (the quantum efficiency \( \eta_{\ell} \) is not important for correlators), and \( \epsilon_{\ell}(t) \) are the uncorrelated white noises with \( \langle \epsilon_{\ell}(t) \epsilon_{\ell}(t') \rangle = \delta_{\ell\ell'} \delta(t - t') \). The Markovian ensemble-averaged evolution of the qubit state is given by Eq. (2) without the noise term,

\[ \dot{r}_{\text{ens}} = \Lambda_{\text{ens}}(r_{\text{ens}} - r_{st}). \]  

(4)
The evolution is assumed to start with some initial state \( r_{\text{in}} \) at time \( t_{\text{in}} \leq t_1 \). All parameters of the measurement and evolution (\( \Lambda_{\text{ens}}, r_{st}, n_{\ell}, \tau_{\ell}, \eta_{\ell} \)) can be time-dependent.

We will first prove the collapse recipe in a simple way and then will prove it in another, more formal way.

1. Simple proof

The simple proof of the collapse recipe closely follows the proof for two-time correlators in Refs. [1] and [2]. Understanding of this proof is easier after understanding of proofs in Refs. [1] and [2].

The proof uses linearity of quantum mechanics. In particular, from the linearity, the correlator (1) can depend on the initial state \( r_{\text{in}} \) only linearly, \( K = v r_{\text{in}} + C \), where the vector \( v \) and the number \( C \) can depend on all parameters for the correlator, but do not depend on \( r_{\text{in}} \). The linearity is better seen by introducing 4-vectors for unnormalized density matrices, \( r = (u, x, y, z) \) for \( \rho = (u \mathbb{1} + x \sigma_x + y \sigma_y + z \sigma_z) / 2 \); then \( K = v r_{\text{in}} \) with some 4-vector \( v \), which does not depend on \( r_{\text{in}} \). Note that quantum evolution is linear for 4-vectors \( \dot{r} \), but is not necessarily linear for 3-vectors \( r \). The evolution, which is linear for 3-vectors \( r \), is called unital.

The correlator (1) is the average over the ensemble of quantum trajectories, starting with initial state \( r_{\text{in}} \) at time \( t_{\text{in}} \). Let us discretize time into small but still non-zero timesteps \( \Delta t \), so that the noises \( \xi_{\ell}(t) \) are not infinitely large \( \langle \xi_{\ell}(t) \rangle \sim 1/\sqrt{\Delta t} \). Since the values of the output signals \( I_{\ell}(t) \) at \( t \neq t_k \) do not affect the correlator \( K \), we can pretend that during these timesteps the signals \( I_{\ell}(t) \) are not available to any observer, and therefore the qubit evolution is equivalent \([3]\) to ensemble-averaged evolution given by Eq. (4). Thus, we need to take into account the full Bayesian evolution (2) only during timesteps \( t_k \). Moreover, since at time \( t_k \) only the output from \( \ell \)-th detector affects the correlator, in Eq. (2) we can neglect all the terms in the sum except for \( \ell = \ell_k \). Integrating Eq. (2) over the timestep \( \Delta t \) around \( t_k \), we obtain the “Bayesian kick”

\[ \Delta r_k \equiv \Delta r(t_k) = \frac{n_{\ell_k} - (n_{\ell_k} \rho_{\ell_k}) r_k}{\sqrt{\tau_{\ell_k}}} \xi_{\ell_k}(t_k) \Delta t, \]  

(5)

where \( r_k \equiv r(t_k) \) and \( |\xi_{\ell_k}| \sim 1/\sqrt{\Delta t} \). With \( \Delta t \to 0 \), this information-induced kick becomes infinitesimally small, so its effect on further evolution is infinitesimally small. However, its contribution to the correlator (1) is significant, since the signal \( I_{\ell_k}(t_k) \) in the correlator contains the term \( \sqrt{\tau_{\ell_k}} \xi_{\ell_k}(t_k) \) [see Eq. (3)], so the effect of the Bayesian kick (5) is proportional to the product

\[ \sqrt{\tau_{\ell_k}} \xi_{\ell_k}(t_k) \Delta r_k = |n_{\ell_k} - (n_{\ell_k} \rho_{\ell_k}) r_k| \xi_{\ell_k}(t_k) \Delta t, \]  

(6)

which is non-zero since on average \( \xi_{\ell_k}^2(t_k) \Delta t = 1 \).

Let us prove that we can apply the collapse recipe to the measurement at time \( t_k \) in the correlator (1). This
means that the value of the correlator would not change if we replace the actual signal \( I_{t_k}(t_k) \) in Eq. (1) by \( I_{t_k}(t_k) = \pm 1 \) with probabilities

\[
p_k^\pm = \frac{1 \pm n_{t_k} r_k}{2}, \tag{7}
\]

and correspondingly greatly increase the Bayesian kick by starting the further evolution with the state \( r(t_k + 0) = \pm n_{t_k} \) (i.e., the corresponding eigenstate of the measured operator \( \sigma_{t_k} \)). In the proof we assume fixed (though arbitrary) values for all previous measurements \( I_{t_k'} \) \((k' < k)\), so that \( r_k \) is fixed. Then the \( N \)-time correlator (1) reduces to a product of \( I_{t_1}(t_1) I_{t_2}(t_2) \ldots I_{t_{k-1}}(t_{k-1}) \) and the remaining \((N+1-k)\)-time correlator. Therefore, this correlator depends linearly on \( r_k \) (better to say, on 4-vector \( r_k \) – see discussion above).

Let us separate the correlator (1) [with fixed \( I_{t_k}(t' < k) \)] into two terms, \( K = K_k^{(1)} + K_k^{(2)} \), which correspond to the two terms in Eq. (3) at time \( t_k \), i.e.,

\[
K_k^{(1)} = \langle I_{t_1}(t_N) \ldots I_{t_{k+1}}(t_{k+1}) \rangle r_k n_{t_k} \text{ and } K_k^{(2)} = \langle I_{t_1}(t_N) \ldots I_{t_{k+1}}(t_{k+1}) \langle \sigma_{t_k} \rangle \rangle I_{t_{k+1}}(t_{k+1}) \rangle \ldots I_{t_k}(t_k) \rangle.
\]

Because of the quantum linearity, the value of \( K_k^{(1)} \) will not change if we replace \( n_{t_k} r_k \) with \( I_{t_k} = +1 \) and start the further evolution with the unnormalized density matrix \( \rho_k^+(t_k + 0) = (n_{t_k} r_k) I_{t_k}(t_k) \rangle / 2 + (n_{t_k} r_k) \rangle / 2 \). Note that we need to multiply all elements of \( \rho_k \) by \( n_{t_k} r_k \); this is why the normalization changes, \( \text{Tr}(\rho_k^+ / 2 + \langle \sigma \rangle / 2) \). This is necessary because for non-zero stationary state \( r_k \), the evolution (4) of the Bloch-sphere components is non-linear, even though the evolution of the density matrix \( \rho_k \) is linear.

The same linearity-based idea for \( K_k^{(2)} \) needs to take into account the Bayesian kick (5). It is easy to see that \( K_k^{(2)} \) will not change if we replace \( \sqrt{\tau_k} \xi_{t_k} \) with \( I_{t_k} = +1 \) and start the further evolution with \( \rho_k^+(t_k + 0) = (n_{t_k} r_k) \rangle / 2 + (n_{t_k} r_k) \rangle / 2 \). Using the linearity again, we see that \( K \) will also not change if we replace \( I_{t_k} \) with \( I_{t_k} = -1 \) and start the further evolution with \( \rho_k^-(t_k + 0) = - (n_{t_k} r_k) \rangle / 2 + (n_{t_k} r_k) \rangle / 2 \). The value of \( K \) will also not change if we use one of these two replacements probabilistically. Note that \( \rho_k^\pm(t_k + 0) \) differ from the normalized eigenstates \( \langle + \rangle / 2 + \langle - \rangle / 2 \) of the measured operator \( \sigma_{t_k} \) only by \( \pm n_{t_k} r_k \rangle / 2 \). If we choose the replacements \( I_{t_k} = \pm 1 \) with probabilities given by Eq. (7), then the effect of this difference will be cancelled on average since \( \langle \pm n_{t_k} r_k \rangle = 0 \). Therefore, the value of \( K \) does not change if we start the further evolution with the states \( \langle + \rangle / 2 + \langle - \rangle / 2 \), as if after the standard projective measurement of \( \sigma_{t_k} \).

Thus, we have proven that we can apply the collapse recipe to the measurement at time \( t_k \), assuming fixed measurement results for the previous measurements. Since the values of the previous measurement results are arbitrary, the assumption of fixed results is not needed. Finally, since the collapse recipe can be applied separately to measurement at any time moment \( t_k \) in the correlator (1), it can be applied to all of them. This completes the proof of the collapse recipe for multi-time correlators (1).

Note that instead of using the collapse recipe and working with normalized states, we can also calculate the correlator using the described above procedure based on unnormalized states. In this case at each moment \( t_k \), we replace \( I_{t_k}(t_k) \) with \( I_{t_k} = +1 \) and start the further evolution with \( \rho_k^+(t_k + 0) = (n_{t_k} r_k) u_k \rangle / 2 + (n_{t_k} r_k) \rangle / 2 \). Note that \( u_k \text{Tr}(\rho_k(t_k)) \) accounts for possibly unnormalized state \( \rho_k \) before \( t_k \). Since this procedure can be applied for all \( N \) moments \( t_k \) and then the product of all \( I_{t_k} \) is 1, the value of the correlator is simply the norm of the state after the last time moment \( t_N \). Therefore this new “one-path recipe” for calculating the \( N \)-time correlator (1) is the following. Start with the initial (normalized) state \( r_{t_0} \) at the initial time \( t_0 \) and propagate it using the ensemble-averaged evolution (4) (which does not change the norm), also adding the “state jumps” (which change the norm) at time moments \( t_k \) as \( \rho_k(t_k) = (n_{t_k} r_k) u_k \rangle / 2 + (n_{t_k} r_k) \rangle / 2 \). Then the norm of the resulting state \( \rho_k(t_N + 0) \) is the value of the correlator.

This one-path recipe can be easily generalized to arbitrary measurement operators in an arbitrary system. For a continuous measurement of an arbitrary Hermitian observable \( A \), the quantum Bayesian evolution due to informational backaction (in the absence of a unitary backaction) is (the derivation is simple, the result is the same as in the Quantum Trajectory theory)\( [4] \)

\[
\dot{\rho} = \frac{ApA - (A^2 \rho + \rho A^2)/2}{2\eta S} + \frac{Ap + \rho - 2\rho \text{Tr}(Ap)}{\sqrt{2S}} \xi(t),
\]

where \( \xi(t) \) is the normalized white noise, \( \langle \xi(t) \rangle = \delta(t - t') \), extracted from the normalized detector signal, \( I(t) = \text{Tr}(Ap) + \sqrt{S/2} \xi(t) \). S is the single-sided spectral density of the detector signal noise, and \( \eta \) is the quantum efficiency (so that the fraction \( \eta \) of the noise \( S \) is “quantum-limited”). Since the first term in Eq. (8) is obviously the ensemble-averaged (Lindblad) evolution, the evolution due to measurement of several (generally non-commuting) observables \( A_k \) in the presence of additional unitary evolution and decoherence (but still without unitary backaction from measurement) is

\[
\dot{\rho} = \mathcal{L}[\rho] + \sum_k \frac{A_k \rho + \rho A_k - 2\rho \text{Tr}(A_k \rho)}{\sqrt{2S}} \xi_k(t),
\]

where \( I_k(t) = \text{Tr}(A_k \rho) + \sqrt{S \xi/2} \xi_k(t), \) \( \langle \xi_k(t) \rangle = \delta(t - t') \), and \( \mathcal{L}[\rho] \) is the ensemble-averaged Lindb-
considered in this paper. A measurement of
This form corresponds to the result
quantum regression formula. Because then
A qubit Pauli operators in an arbitrary system of qubits,
becomes obscure.) In particular, the collapse recipe is
alize the collapse recipe to work with normalized, but
in the general case is
\begin{align}
\rho(t_k + 0) = (A_{t_k} \rho_{t_k} + \rho A_{t_k})/2.
\end{align}

Thus, the one-path recipe for the N-time correlator (1)
where \(E(t|t')\) is the trace-preserving ensemble-averaged
evolution (operation) from time \(t'\) to \(t\) due to Lindblad
term \(\dot{\rho} = L[\rho]\), while \(M_{t_k} \rho = (A_{t_k} \rho + \rho A_{t_k})/2\) is the
trace-changing operation, related to measurement (with-
out unitary backaction) of the operator \(A_{t_k}\) at time \(t_k\). If
a unitary backaction of the form \(\sum_x e^{-i[B_{t_k}, \rho] \xi(t)} / \sqrt{2\pi t}\)
is added into Eq. (9) \((B_t\) are Hermitian), with the contribution
to the ensemble-averaged evolution absorbed by
\(L[\rho]\), then the additional Bayesian kick produces an extra
term in Eq. (10): \(M_{t_k} \rho = (A_{t_k} \rho + \rho A_{t_k})/2 - i[B_{t_k}, \rho]/2\).
The one-path recipe is similar to the result of a recent
paper \cite{5} by Tilloy. Note the similarity of Eq. (11) to the
quantum regression formula.

The one-path recipe (11) based on unnormalized states
can be reduced to the physically transparent collapse
recipe (based on physical states) only when \(B_t = 0\)
and \(A_{t_k}^2\) are positive numbers, i.e., scaled unity opera-
tors. (In the general case, it is still possible to genera-
ize the collapse recipe to work with normalized, but
unphysical states; however, then the physical meaning
becomes obscure.) In particular, the collapse recipe is
fully applicable for continuous measurement of multi-
qubit Pauli operators in an arbitrary system of qubits,
because then \(A_{t_k}^2 = 1\). One can see this by noticing
that Eq. (10) in this case can be written as \(\rho(t_k + 0) =
\sum_{\pm} \pm \text{Tr}[\rho_{t_k} \Pi_{t_k}^\pm] \left( \frac{\Pi_{t_k}^\pm \rho_{t_k} \Pi_{t_k}^\pm}{\text{Tr}[\rho_{t_k} \Pi_{t_k}^\pm]} \right)\), where \(\Pi_{t_k}^\pm\) is the projection
operator corresponding to the eigenvalue \(\pm 1\) of \(A_{t_k}\).
This form corresponds to the result \(\pm 1\) of the projective
measurement of \(A_{t_k}\), with probability \(\text{Tr}[\rho_{t_k} \Pi_{t_k}^\pm]\) and with
the density matrix inside the parenthesis being the nor-
malized state after the projective multi-qubit collapse.

Completing the brief digression into the general case,
we remind that the main purpose of this section is the
proof of the collapse recipe for the case of a single qubit,
considered in this paper.

2. Alternative proof

Now let us prove the collapse recipe for the single-qubit
state in a different, more formal way. In this derivation
we will also obtain the correlator factorization result for
unital evolution, Eq. (11) of the main text.

In addition to the correlator \(K\) given by Eq. (1), let us
introduce the vector-valued correlator
\begin{align}
K_{t_1,...,t_N}(t_1,...,t_N) = \langle r(t_N) I_{t_{N-1}}(t_{N-1}) \cdots I_{t_1}(t_1) \rangle.
\end{align}

Note that in this notation for \(K\), the last subscript \(t_N\)
is not needed, but we keep it to remind us that \(K\) is
an average product of \(N\) terms. We will usually assume
t_1 < t_2 < \cdots < t_N (as for the correlator \(K\)), but at some
point in the derivation we will need the time moment \(t_N\)
to cross \(t_{N-1}\). The correlator \(K\) can be easily obtained
from \(K\) as
\begin{align}
K_{t_1,...,t_N}(t_1,...,t_N) = n_{t_N} K_{t_1,...,t_N}(t_1,...,t_N),
\end{align}
since the noise contribution \(\sqrt{t_{N-1}} \xi_{t_{N-1}}(t_N)\) to the output
signal \(I_{t_{N-1}}(t_N)\) [see Eq. (3)] is not correlated with past
qubit states.

Let us separate \(K\) into two terms, \(K = K^{(1)} + K^{(2)}\),
which correspond to the two terms in Eq. (3) for the
signal \(I_{t_{N-1}}(t_{N-1})\),
\begin{align}
K^{(1)}_{t_1,...,t_N}(t_1,...,t_N) &\equiv \langle r(t_N) n_{t_{N-1}} r(t_{N-1}) \rangle \\
&\times I_{t_{N-2}}(t_{N-2}) \cdots I_{t_1}(t_1),
\end{align}
\begin{align}
K^{(2)}_{t_1,...,t_N}(t_1,...,t_N) &\equiv \langle r(t_N) \sqrt{t_{N-1}} \xi_{t_{N-1}}(t_{N-1}) \rangle \\
&\times I_{t_{N-2}}(t_{N-2}) \cdots I_{t_1}(t_1).
\end{align}
The derivative of \(K^{(1)}\) over the last time moment \(t_N\) can be
obtained from Eq. (2),
\begin{align}
\partial_{t_N} K^{(1)}_{t_1,...,t_N}(t_1,...,t_N) &\equiv \Lambda_{\text{ens}}(t_N) \left( K^{(1)}_{t_1,...,t_N}(t_1,...,t_N) \\
&- r_{st}(t_N) K_{t_1,...,t_{N-1}}(t_1,...,t_{N-1}) \right),
\end{align}
where we included possible dependence of \(\Lambda_{\text{ens}}\) and \(r_{st}\)
on time. The initial condition at \(t_N = t_{N-1} + 0\) is
\begin{align}
K^{(1)}_{t_1,...,t_N}(t_1,...,t_{N-1},t_{N-1} + 0) = \langle r(t_{N-1}) \rangle \\
&\times (n_{t_{N-1}} r(t_{N-1})) I_{t_{N-2}}(t_{N-2}) \cdots I_{t_1}(t_1).
\end{align}
The time derivative of \(K^{(2)}\) over \(t_N\) can also be obtained
from Eq. (2), which gives
\begin{align}
\partial_{t_N} K^{(2)}_{t_1,...,t_N}(t_1,...,t_N) &\equiv \Lambda_{\text{ens}}(t_N) K^{(2)}_{t_1,...,t_N}(t_1,...,t_N) \\
&+ \left( n_{t_{N-1}} - r(t_{N-1}) (n_{t_{N-1}} r(t_{N-1})) \right) \\
&\times I_{t_{N-2}}(t_{N-2}) \cdots I_{t_1}(t_1) \delta(t_{N-1} - t_{N-1}).
\end{align}
Note that \(K^{(2)} = 0\) for \(t_N < t_{N-1}\) because of causality,
so the second term in Eq. (18) sets the initial condition
at \( t_N = t_{N-1} + 0 \), caused by the Bayesian kick. Also note that at \( t_N > t_{N-1} \), the evolution of \( K^{(2)} \) is linear (due to \( \Lambda_{\text{ens}} \)); it does not have the inhomogeneous term containing \( r_{\text{st}} \) as for the evolution of \( K^{(1)} \) in Eq. (16).

Solving Eqs. (16)–(18), we find \( K^{(1)} \) and \( K^{(2)} \) for \( t_N > t_{N-1} \), starting with the value (17) of \( K^{(1)} \) at \( t_N = t_{N-1} + 0 \):

\[
K_{t_1,\ldots,t_N}^{(1)}(t_1,\ldots,t_N) = \mathcal{P}(t_N|t_{N-1}) K_{t_1,\ldots,t_N}^{(1)}(t_1,\ldots,t_{N-1} + 0) + \mathcal{P}_{\text{st}}(t_N|t_{N-1}) K_{t_1,\ldots,t_N}^{(1)}(t_1,\ldots,t_{N-1}),
\]

\[
K_{t_1,\ldots,t_N}^{(2)}(t_1,\ldots,t_N) = -\mathcal{P}(t_N|t_{N-1}) K_{t_1,\ldots,t_N}^{(1)}(t_1,\ldots,t_{N-1} + 0) + \mathcal{P}(t_N|t_{N-1}) n_{t_{N-1}}K_{t_1,\ldots,t_{N-2}}(t_1,\ldots,t_{N-2}),
\]

where \( \mathcal{P}(t|t') \) is the 3×3 propagator matrix for the homogeneous part of the ensemble-averaged evolution (4), so that \( \partial_t \mathcal{P}(t|t') = \Lambda_{\text{ens}}(t) \mathcal{P}(t|t') \) for \( t > t' \) and \( \mathcal{P}(t|t) = \mathbb{1} \), while \( \mathcal{P}_{\text{st}}(t|t') \) is the contribution from the inhomogeneous part:

\[
\mathcal{P}_{\text{st}}(t|t') = -\int_{t'}^t \mathcal{P}(t|t'') \Lambda_{\text{ens}}(t'') r_{\text{st}}(t'') \, dt''.
\]

From Eqs. (19)–(21) and (13)–(15) we find the recursive formula, which relates the \( N \)-time correlator \( K_{t_1,\ldots,t_N}(t_1,\ldots,t_N) \) with \( (N-1) \)-time correlator and \( (N-2) \)-time correlator for \( N > 2 \):

\[
K_{t_1,\ldots,t_N}(t_1,\ldots,t_N) = n_{t_N} \mathcal{P}(t_N|t_{N-1}) n_{t_{N-1}}
\times K_{t_1,\ldots,t_{N-2}}(t_1,\ldots,t_{N-2}) + n_{t_N} \mathcal{P}_{\text{st}}(t_N|t_{N-1}) K_{t_1,\ldots,t_{N-1}}(t_1,\ldots,t_{N-1}).
\]

Note that for \( N = 2 \), the only difference in the derivation is that the product \( I_{t_{N-2}}(t_{N-2}) \cdots I_{t_1}(t_1) \) in Eq. (18) should be replaced with 1. As a consequence, the \( (N-2) \)-time correlator in Eqs. (20) and (22) should be replaced with 1. Therefore, Eq. (22) is also valid for \( N = 2 \) if we define the 0-time correlator as being equal to 1.

Thus, the recursive relation (22) is sufficient to derive explicit formulas for \( N \)-time correlators, if we complement it with the correlator for \( N = 1 \), which is simple:

\[
K_{t_1}(t_1) = n_{t_1} r(t_1).
\]

Now let us show that the \( N \)-time correlators obtained via Eqs. (22) and (23) coincide with the correlators obtained using the collapse recipe. Since for \( N = 1 \) the collapse recipe obviously gives Eq. (23), we only need to prove that the recursive relation (22) also follows from the collapse recipe (with the correlator for \( N = 0 \) defined as 1). Note that applicability of the collapse recipe to the two-time correlator was proven in Ref. [2].

Let us rewrite Eq. (7) of the main text (following from the collapse recipe, as indicated by the superscript below) in the form

\[
K_{t_1,\ldots,t_N}^{(\text{coll})}(t_1,\ldots,t_N) = \sum_{\{I_k = \pm 1\}} I_N
\times \frac{1}{2} \prod_{k=2}^{N-1} \left[ I_k p(I_k,t_k|I_{k-1},t_{k-1}) \right] \times I_{t_1} p(I_{t_1},t_1),
\]

where

\[
r_{\text{ens}}(t_N, I_{t_N-1}, t_{N-1}) = I_{t_N-1} \mathcal{P}(t_N|t_{N-1}) n_{t_{N-1}}
+ \mathcal{P}_{\text{st}}(t_N|t_{N-1}),
\]

is the solution of the ensemble-averaged evolution (4) with the initial condition \( I_{t_N-1}, n_{t_{N-1}} \) at time \( t_{N-1} \). Note that the last line of Eq. (24) summed over all combinations of \( I_{t_k} = \pm 1 \) except summation over \( I_{t_N} \), is the \( (N-1) \)-time correlator \( K_{t_1,\ldots,t_{N-1}}^{(\text{coll})}(t_1,\ldots,t_{N-1}) \).

The term 1 in the second line of Eq. (24) can be removed because of summation over \( I_{t_N} = \pm 1 \). After removing 1, we see that \( I_{t_N} \) in the first and second lines cancel each other since \( I_{t_N}^2 = 1 \). Therefore, Eq. (24) can be rewritten as

\[
K_{t_1,\ldots,t_N}^{(\text{coll})}(t_1,\ldots,t_N) = \sum_{\{I_k = \pm 1\}} n_{t_N}
\times \left[ I_{t_{N-1}} \mathcal{P}(t_N|t_{N-1}) n_{t_{N-1}} + \mathcal{P}_{\text{st}}(t_N|t_{N-1}) \right]
\times \prod_{k=2}^{N-1} \left[ I_k p(I_k,t_k|I_{k-1},t_{k-1}) \right] \times I_{t_1} p(I_{t_1},t_1),
\]

where there is already no summation over the last output \( I_{t_N} \), and we used Eq. (25) for \( r_{\text{ens}} \). Let us separate \( K^{(\text{coll})} \) into two terms, corresponding to contributions from the two terms in the second line of Eq. (26). The second term (containing \( \mathcal{P}_{\text{st}} \)) is \( n_{t_N} \mathcal{P}_{\text{st}}(t_N|t_{N-1}) K_{t_1,\ldots,t_{N-1}}^{(\text{coll})}(t_1,\ldots,t_{N-1}) \), thus coinciding with the third line of Eq. (22). In the remaining first term, let us substitute the product \( \prod_{k=2}^{N-2} \) multiplied by the corresponding factor for \( k = N - 1 \), and then use relations \( I_{t_N}^2 = 1 \) and \( \sum_{I_{t_N} = \pm 1} p(I_{t_{N-1}},t_{N-1}|I_{t_{N-2}},t_{N-2}) = 1 \). This gives us

\[
\mathcal{P}_{\text{st}}(t_N|t_{N-1}) n_{t_{N-1}} K_{t_1,\ldots,t_{N-2}}^{(\text{coll})}(t_1,\ldots,t_{N-2}),
\]

which is the first term in Eq. (22). Thus, we have obtained the same recursive relation (22) for \( K^{(\text{coll})} \). Therefore, we have proven that the collapse recipe gives the same result for \( N \)-time correlators as the calculation based on the stochastic evolution equation (2).

Note that the recursive relation (22) can be used directly to derive the main result of the paper: factorization of the \( N \)-time correlator in the case of unital evolution, Eq. (11) of the main text. Since \( r_{\text{st}} = 0 \) for unital evolution, from Eq. (21) we obtain \( \mathcal{P}_{\text{st}} = 0 \), so the recursive formula (22) relates the \( N \)-time correlator only with the \( (N-2) \)-time correlator. It is easy to see that the coefficient \( n_{t_N} \mathcal{P}(t_N|t_{N-1}) n_{t_{N-1}} \) is the two-time correlator \( K_{t_{N-1},t_N}(t_{N-1},t_N) \), as also follows from Eq. (22) for \( N = 2 \), since \( K = 1 \) for \( N = 0 \). Thus, for unital evolution, the \( N \)-time correlator is a product of two-time correlator for the two latest time moments and the remaining \( (N-2) \)-time correlator. This gives Eq. (11) of the main text.
Section B: Experimental multi-time correlators

Our theoretical results for the correlators have been checked against experimental data from the experiment, in which a physical qubit (transmon), embedded into a 3D Al cavity, is subject to relatively fast Rabi oscillations with frequency $\Omega_R/2\pi = 40$ MHz. The physical qubit is dispersively coupled to the two lowest cavity modes; each of them is off-resonantly driven with two sideband tones at frequencies $\omega_{\delta i} = \Omega_R + \Omega_L$ (where $\Omega_L = \Omega_l R$), with a relative phase $\delta_i$. Here $i = x, y$ labels the cavity mode that performs continuous measurement of the observable $\sigma_i$ of the effective (rotating frame) qubit, and $\omega_{\delta i}$ is the frequency of the corresponding cavity mode. Details of the measurement technique are discussed in Ref. [6] (see also Ref. [2]).

The measured effective qubit is defined in the frame, which rotates with frequency $\Omega_R$ with respect to the laboratory frame of the physical qubit. The measurement axes on the Bloch sphere of the effective qubit are determined by the relative phases $\delta_i$ of the sideband tones, with position of the effective z axis defined arbitrarily within the $xz$ plane of the physical qubit rotations. We choose one of the measurements to be exactly the $\sigma_z$ measurement; the other measurement direction is shifted by an angle $\varphi$, thus corresponding to the observable $\sigma_\varphi \equiv \sigma_z \cos \varphi + \sigma_x \sin \varphi$. In the experiment $\varphi = n\pi/10$ with integer $n = 0, 1, 2, \ldots, 10$. The effective qubit is initialized at $t = 0$ in the middle between the measurement axes, i.e., at the states $r_0^x, y = \pm \{\sin(\varphi/2), 0, \cos(\varphi/2)\}$. Approximately 200,000 readout trajectories are recorded for each angle $\varphi$, with approximately 100,000 trajectories for each initial state (we use only trajectories, selected by heralding the ground state at the start of a run and checking that transmon is still within the two-level subspace after the run [2]).

The ensemble-averaged evolution for the effective qubit is [2]

\begin{align}
\dot{x} &= -\Gamma_x x - \Gamma_\varphi \cos \varphi (x \cos \varphi - z \sin \varphi) + \tilde{\Omega} z - \gamma x, \\
\dot{y} &= -(\Gamma_z + \Gamma_\varphi + T_2^{-1}) y, \\
\dot{z} &= \Gamma_\varphi \sin \varphi (x \cos \varphi - z \sin \varphi) - \tilde{\Omega} x - \gamma z, 
\end{align}

where $\Gamma_z$ and $\Gamma_\varphi$ are the measurement-induced dephasing rates in the corresponding bases of the two measurement channels as

$$K_{\varphi z \varphi} (\Delta t_{21}, \Delta t_{32}) = \int^{t_a+T}_{t_a} \left\langle \frac{\tilde{I}_x (t + \Delta t_{21} + \Delta t_{32}) - \tilde{I}_{\text{off}}}{\Delta \tilde{I}_x/2} \cdot \frac{\tilde{I}_z (t + \Delta t_{21}) - \tilde{I}_{\text{off}}}{\Delta \tilde{I}_z/2} \right\rangle \, dt / T,$$

with the time integration over duration $T = 0.2 \mu s$ is needed to reduce correlator fluctuations, the small constant offsets $\tilde{I}_{\text{off}}$ are less than 0.2 in magnitude (see [2] for details), and experimental responses are $\Delta \tilde{I}_z = 4.2$ (in Ref. [2] we used 4.0) and $\Delta \tilde{I}_x = 4.4$. To avoid initial transients in the data, we use $t_a = 1 \mu s$. The superscripts in the correlators $K_{\varphi' \varphi''}$ correspond to the initial states $r_0^\pm$, the ensemble averaging is over the corresponding $\sim 100,000$ trajectories. Since theoretically $K_{\varphi' \varphi''} = -K_{\varphi'' \varphi'}$, in Fig. 1 of the main text we plot the difference,

$$K_{\varphi z \varphi} (\Delta t_{21}, \Delta t_{32}) = \left[ K_{\varphi z \varphi} (\Delta t_{21}, \Delta t_{32}) - K_{\varphi' z \varphi'} (\Delta t_{21}, \Delta t_{32}) \right] / 2. \quad (31)$$

The experimental four-time correlators plotted in Fig. 2 of the main text are calculated as

$$K_{\varphi z \varphi z} (\Delta t_{21}, \Delta t_{32}, \Delta t_{43}) = \int^{t_a+T}_{t_a} \left\langle \frac{\tilde{I}_x (t + \Delta t_{21}) - \tilde{I}_{\text{off}}}{\Delta \tilde{I}_x/2} \cdot \frac{\tilde{I}_z (t + \Delta t_{31}) - \tilde{I}_{\text{off}}}{\Delta \tilde{I}_z/2} \right\rangle \, dt / T,$$

$$K_{\varphi z \varphi z} (\Delta t_{21}, \Delta t_{32}, \Delta t_{43}) = \int^{t_a+T}_{t_a} \left\langle \frac{\tilde{I}_x (t + \Delta t_{21}) - \tilde{I}_{\text{off}}}{\Delta \tilde{I}_x/2} \cdot \frac{\tilde{I}_z (t + \Delta t_{31}) - \tilde{I}_{\text{off}}}{\Delta \tilde{I}_z/2} \right\rangle \, dt / T,$$

where $\Delta t_{31} = \Delta t_{32} + \Delta t_{21}, \Delta t_{41} = \Delta t_{43} + \Delta t_{31}$, the averaging is now over all trajectories (starting from both $r_0^x$ and $r_0^y$), and we use $T = 500 \mu s$ and $t_a = 1 \mu s$. We need a larger averaging window $T$ since the four-time correlators are noisier than the three-time correlators.

The theoretical lines in Figs. 1 and 2 of the main text are calculated using the two-time correlator $K_{\varphi z \varphi} (\tau) = K_{\varphi z \varphi} (\tau_1, \tau_1 + \tau)$ derived in Ref. [2] (see below); for the three-time correlator we also need the average $\langle I_x (t) \rangle = z(t) \cos \varphi + x(t) \sin \varphi$, which is calculated using Eqs. (27)–(29) with the initial condition $r(0) = r_0^x$. The two-time correlator $K_{\varphi z \varphi} (\tau)$ is calculated analytically as [2]

$$K_{\varphi z \varphi} (\tau) = \frac{(\Gamma_z + \Gamma_\varphi) \cos \varphi + 2\tilde{\Omega} \sin \varphi}{2(\Gamma_z + \Gamma_\varphi)} (e^{-\Gamma_z \tau} - e^{-\Gamma_\varphi \tau})$$

$$+ \frac{\cos \varphi}{2} (e^{-\Gamma_z \tau} + e^{-\Gamma_\varphi \tau}), \quad (33)$$

where

$$\Gamma_\pm = \Gamma_z + \Gamma_\varphi \pm \frac{[\Gamma_z^2 + \Gamma_\varphi^2 + 2\Gamma_z \Gamma_\varphi \cos (2\varphi) - 4\tilde{\Omega}^2]^{1/2}}{2}. \quad (34)$$