Analysis of the radio-frequency single-electron transistor with large quality factor

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We have analyzed the response and noise-limited sensitivity of the radio-frequency single-electron transistor (rf-SET), extending the previously developed theory to the case of arbitrary large quality factor $Q$ of the rf-SET tank circuit. It is shown that while the rf-SET response reaches the maximum at $Q$ roughly corresponding to the impedance matching condition, the rf-SET sensitivity worsens monotonically with the increase of $Q$. In addition, we propose an operation mode in which an overtone of the incident rf wave is in resonance with the tank circuit. © 2003 American Institute of Physics. [DOI: 10.1063/1.1614840]

The problem of relatively small bandwidth of the conventional single-electron transistor (SET) due to its large output resistance, has been solved for many applications by the invention of the radio-frequency SET (rf-SET), which in many instances has already replaced the traditional SET setup. The principle of the rf-SET operation is somewhat similar to the operation of the rf superconducting quantum interference device and is based on the microwave reflection from a tank circuit containing the SET (Fig. 1), which affects the quality factor ($Q$-factor) of the tank; another possibility is to use the transmitted wave. The wide bandwidth of the rf-SET is due to the signal propagation by the microwave instead of charging the output wire, while the tank circuit provides a better match between the cable wave impedance $R_0=50\,\Omega$ and much larger SET resistance ($\sim 10^5\,\Omega$).

The rf-SET bandwidth over 100 MHz has been demonstrated using the microwave carrier frequency $\omega/2\pi =1.7$ GHz and relatively low $Q$-factor $Q=6$. However, in the present-day experiments, the bandwidth is typically about 10 MHz because of lower carrier frequency (to reduce amplifier noise) and higher $Q$-factor (as an example, the bandwidth of 7 MHz for $\omega/2\pi=332$ MHz and $Q>20$ has been reported in Ref. 6). Since the SET sensitivity is limited by the $1/f$ noise only at frequencies below a few kilohertz, the rf-SET typically operates in the frequency range of shot-noise-limited sensitivity of the SET. The rf-SET charge sensitivity of $3.2\times 10^{-6}\,e/\sqrt{\text{Hz}}$ (4.8$\hbar$ in energy units) at 2 MHz has been reported in Ref. 6 (this value still contains comparable contributions from the SET shot noise and amplifier noise). Such sensitivity and bandwidth are almost enough for a single-shot readout of a Cooper-pair-box qubit.

In spite of significant experimental rf-SET activity, we are aware of only few theoretical papers on the rf-SETs. The basic theory of the shot-noise-limited sensitivity of the rf-SET has been developed in Ref. 13. A similar theory has been applied to the analysis of the sensitivity of the rf-SET-based micromechanical displacement detector. Some theoretical analysis of the transmission-type rf-SET can be found in Ref. 10.

In this letter, we extend the theory of Ref. 13 to the case of arbitrary large $Q$-factor of the tank circuit, removing the assumption (violated in the present-day experiments) of $Q$ being much smaller than the impedance-matching value. We calculate the response and sensitivity of the normal-metal rf-SET and find the optimal values numerically. Besides the usual case of the carrier wave in resonance with the tank circuit, we also consider the regime of a resonant overtone and find a comparable rf-SET performance in this case.

We consider a SET (Fig. 1) consisting of two tunnel junctions with capacitances $C_{1j}$ and $C_{2j}$ and resistances $R_1$ and $R_2$. The measured charge source $q_S$ has the capacitance $C_S=C_{S1}+C_{S2}$ and is coupled to the SET via capacitance $C_g$. Assuming constant $q_S$ (neglecting back-action), the SET can be reduced to the effective double-junction SET with capacitances $C_1=C_{1j}+C_gC_{S1}/(C_g+C_S)$, $C_2=C_{2j}+C_gC_{S2}/(C_g+C_S)$ and background charge $q_0=q_{00}+q_gC_g/(C_g+C_S)$, where $q_{00}$ is the initial contribution. We will calculate the rf-SET response and sensitivity in respect to $q_g$, while the corresponding quantities in respect to the measured charge $q_S$ differ by the factor $C_g/(C_g+C_S)$.

The current $I(t)$ through the SET affects the quality factor of the tank circuit consisting of the capacitance $C_T$ and inductance $L_T$. In the linear approximation, the SET can be replaced by an effective resistance $R_d$, and the total (“loaded”) quality factor $Q_L=(1/Q+1/Q_{SET})^{-1}$ has contributions from the “unloaded” $Q$-factor $Q=\sqrt{Z}\sqrt{L_T/C_T}/R_0$ and damping by the SET: $Q_{SET}=R_d/\sqrt{L_T/C_T}$. For the incoming voltage wave $\hat{V}_i\exp(i\omega t)$, the reflected wave $\alpha \hat{V}_i\exp(i\omega t)$ depends on the reflection coefficient $\alpha=(Z-R_0)/(Z+R_0)$, where $Z=\frac{i\omega L_T+(\omega C_T+1/R_d)}{\omega C_T+1/R_d}$; close to the resonance, $\omega=\omega_0=(L_TC_T)^{-1/2}$, it can be approximated as $Z=\frac{L_T/C_T R_d+2i(L_T/C_T)^{1/2}\Delta\omega/\omega_0}{\omega^2}$, where $\Delta\omega=\omega-\omega_0$. Since an increment of the measured charge $q_S$ leads to an increment

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of \( R_d \), the rf-SET response is proportional to \( d\alpha/dR_d \). However, the amplitude \( \dot{V}_b = \dot{V}_b(Z+R_d)(i\omega Ct_1+1/R_d)/2 \) of the SET bias voltage oscillations should be determined by the Coulomb blockade threshold; so a more representative quantity is

\[
\frac{d\alpha}{dR_d} \dot{V}_b = -iR_0 \frac{Q}{1 + Q^2 R_0/R_d} \frac{1}{1 + 2iQ \Delta \omega / \omega_0}. \tag{1}
\]

This equation shows that the rf-SET response reaches the maximum at \( Q = (R_d / R_0)^{1/2} \), which is the case of matched impedances at resonance, \( Z = R_0 \), and corresponds to the condition \( Q = Q_{SET} = 2Q_L \).

The linear analysis can only be used as an estimate because of the significant nonlinearity of the SET \( I-V \) dependence. For the full analysis, we use the differential equation

\[
\dot{v} / \omega_0^2 + \dot{v} / Q \omega_0 + v = 2(1 - \omega_0^2 / \omega_0^2) V_{in} \cos \omega t - R_d \dot{I}(t) - \ddot{I}(t),
\]

where \( v(t) = V_{in} \cos \omega t + \dot{v}_0(t) \) is the voltage at the end of the cable with subtracted dc component \( V_0 \) (see Fig. 1; do not use complex representation any more). The current \( I(t) \) and its dc value \( \langle I \rangle \) are found self-consistently from the SET bias voltage \( V_{b}(t) = V_0 + v(t) + [2V_{in} \omega \sin \omega t + \ddot{v}(t)]Q \omega_0 \), using the “orthodox” model \(^1\) and assuming continuous SET current \( (\omega \ll \omega_0) \).

In the steady state, the reflected wave can be represented as

\[
V_{ou}(t) = -V_{in} \cos \omega t + \sum_{n=1}^{\infty} [X_n \cos n \omega t + Y_n \sin n \omega t] \sin \frac{\omega t}{\omega_0}.
\]

where \( X_n \) and \( Y_n \) are the following coefficients \(^2\):

\[
X_n = \{R_0 Q[n - \omega a_n - (1 - \omega_0^2) b_n] + 2Q^2(1 - \omega_0^2)\},
\]

\[
Y_n = \{-R_0 Q[n - \omega b_n + (1 - \omega_0^2) a_n] + 2Q(1 - \omega_0^2)\},
\]

\[
a_n = 2\langle I(t) \sin n \omega t \rangle, \quad b_n = 2\langle I(t) \cos n \omega t \rangle,
\]

where \( \omega = \omega / \omega_0 \), \( \delta_1 \) is the Kronecker symbol, and averaging is over the oscillation period, while \( I(t) \) is determined by the SET voltage \( V_{b}(t) = V_0 + 2Q \omega_0 V_{in} \sin \omega t + \sum_{n=1}^{\infty} [X_n + \omega_0 Y_n \cos n \omega t + (Y_n - \omega_0 \delta_n) \sin n \omega t] \sin \frac{\omega t}{\omega_0} \). Notice that the linear approximation corresponds to neglecting the contribution of overtones \((n \gg 2)\); then \( R_d = \pi A [\int_{-\infty}^{\infty} I(V_0 + A \sin x) \sin x \ dx] \), where \( A \) is the amplitude of \( V_b \) oscillations, \( V_{b}(t) = V_0 + A \sin(\omega t + \delta) \), while there is no effective reactance contribution. We used the self-consistent linear approximation as a starting point for the iterative solution of Eqs. (2)-(4).

The rf-SET response in respect to monitoring the quadrature component \( X_n \) can be defined as a derivative \( dX_n / dq_0 \) (similarly, \( dY_n / dq_0 \) for \( Y_n \) monitoring). Other experimentally relevant definitions are for monitoring the optimized phase-shifted combination \( X_n \cos \phi + Y_n \sin \phi \) and/or the reflected wave amplitude; however, in the cases considered subsequently there is only one leading quadrature, so that different definitions practically coincide.

The corresponding noise-limited sensitivity (minimal detectable charge for the measurement bandwidth \( \Delta f \)) is defined as \( \delta q_0 = |\Delta X| / \sqrt{S_X} = \sqrt{|\Delta Y| / \Delta q_0} \) (similarly, \( \sqrt{|\Delta Y| / \Delta q_0} \)), where the low-frequency spectral densities \( S_Xn \) and \( S_Yn \) of quadrature fluctuations are

\[
S_Xn = c_n^2 \langle S_I(t) \sin^2 n \omega t \rangle + d_n^2 \langle S_I(t) \cos^2 n \omega t \rangle + c_n d_n \langle S_I(t) \sin 2n \omega t \rangle, \tag{5}
\]

\[
S_Yn = d_n^2 \langle S_I(t) \sin^2 n \omega t \rangle + c_n^2 \langle S_I(t) \cos^2 n \omega t \rangle + c_n d_n \langle S_I(t) \sin 2n \omega t \rangle, \tag{6}
\]

where \( c_n = (2R_0 Q \omega_0) \left[ (n^2 \omega_0^2 + Q^2(1 - n^2 \omega_0^2))^2 \right] , \]

\( d_n = c_n Q \left( 1 - n^2 \omega_0^2 \right) / n \omega_0 \), \( S_I(t) \) is the low-frequency spectral density of the SET shot noise \(^2\) with the time dependence due to oscillating bias voltage, and the averaging is over the period \( 2\pi / \omega \).

Figure 2 shows the numerically calculated rf-SET response and sensitivity as functions of the “unloaded” \( Q \)-factor for a symmetric SET \(^1\): \( C_1 = C_2 = C_{S} \), \( R_1 = R_2 = R_S \), with \( R_S = 100 \, k\Omega \) at temperature \( T = 0.01 e^2 / C_S \) for the case of resonant carrier frequency, \( \omega = \omega_0 \). Both the response and sensitivity are shown with respect to the quadrature \( X_1 \) since all other components are small. The rf-SET performance is optimized over the wave amplitude \( V_{in} \) and the charge \( q_0 \) to provide either maximum response (MR mode; solid lines) or optimized sensitivity (OS mode; dashed lines). \(^1\) We show the results for two values of the dc bias voltage \( V_0 \). The case \( V_{in} = 0 \) provides the best MR response and the best OS sensitivity, and corresponds to the symmetric SET operation with respect to positive and negative bias voltages. The other value shown, \( V_{in} = 0.5 e / C_S \), represents a typical case when only one branch of the SET \( I-V \) curve is used, and corresponds to the plateau-like region \(^3\) of the response and sensitivity dependences on \( V_0 \).

As seen from Fig. 2(a), the maximum rf-SET response is achieved at \( Q \)-factors (somewhat different in different regimes) comparable to the rough estimate \((R_S / R_0)^{1/2} \approx 45 \) (although this maximum does not actually correspond to the minimum of reflection). In contrast to the response behavior, the rf-SET sensitivity [Fig. 2(b)] worsens monotonically with \( Q \). Qualitatively, this happens because the noise \( S_{X1} \) in Eq. (5) is proportional to \( Q^2 \), while the response does not grow as fast as \( Q \). At low \( Q \), the OS sensitivity is fitted well by the analytical result \(^3\) \( \delta q_{0 OS} = 2.65 e (R_S / C_{S} \Delta f)^{1/2} \left( T C_S / e^2 \right)^{1/2} \) for \( V_{in} = 0 \), and \( \delta q_{0 OS} = 3.34 C_S (R_S / \Delta f)^{1/2} \) for the asymmetric operation (shown by dotted lines). However, at realistic \( Q \)-factors, \( \delta q_0 \) is significantly larger (by about 50\% at \( Q = 50 \) for data in Fig. 2). Another interesting observation from Fig. 2 is that the response in the MR mode is only moderately \((-30\%) \) better than in the OS mode. These findings show that in order to improve experimental rf-SET sensitivity, it is preferable to use smaller \( Q \)-factor (at least not exceeding the
there is no second overtone because of the sensitivity to overtone becomes significant if responses are much smaller and therefore monitoring of relationships with respect to overtones are comparable to the contribution of overtones in this case is small because they are tantamount overtone is comparable to the performance in the conventional regime. The rf-SET response and sensitivity in the MR and OS modes for several $Q$-factors, $V_0 = 0$, $R_x R_0 = 2000$, and $\omega = \omega_0$.

Figure 3 shows the temperature dependence of the rf-SET response and sensitivity in the MR and OS modes. Even though the low-$T$ analytical formula for the OS sensitivity (mentioned previously) works well only for small $Q$, the $T^{1/2}$ dependence at $T \leq 0.05e^2/C_S$ remains valid for large $Q$-factors (at very small $T$, the OS sensitivity is limited by the neglected here contribution from cotunneling processes\cite{5,18}). The OS response practically does not depend on temperature at $T \leq 0.05e^2/C_S$. The performance in the MR mode saturates below $T = 0.03e^2/C_S$

So far, we have been considering the usual case $\omega = \omega_0$. In spite of significant SET $I-V$ nonlinearity (the SET nonlinearity has been recently used\cite{19} for rf mixing), the contribution of overtones in this case is small because they are off resonance. Even though the formally calculated sensitivities with respect to overtones are comparable to the $X_1$ sensitivity (worse by less than two times for $n=2$ and $3$), the responses are much smaller and therefore monitoring of overtones is impractical. However, the contribution of $n$th overtone becomes significant if $\omega = \omega_0/2$, and $\omega = \omega_0/3$, in respect to monitoring $Y_2$ and $Y_3$, correspondingly (the $X$-quadratures are small). We use $V_0=0$ in the case $\omega = \omega_0/3$ and $V_0=0.5e^2/C_S$ in the case $\omega = \omega_0/2$ (for $V_0=0$ there is no second overtone because of the $I-V$ curve symmetry).

Comparing Figs. 2 and 4 (the parameters are the same), we see that the rf-SET performance in the regime of a resonant overtone is comparable to the performance in the conventional regime $\omega = \omega_0$ (the MR response and OS sensitivity are worse by about 1.5 times).\cite{20} On the other hand, the frequency separation between the incident wave and monitored reflected wave may be an important advantage for some applications. In addition, it may be advantageous to have the absence of the monitored wave when the SET is off (no current), while for the conventional mode, this case corresponds to the largest reflected power. The disadvantage is a larger incident wave amplitude $V_n$, than for a conventional rf-SET regime, which may lead to heating problems. Nevertheless, we hope that the proposed mode of the resonant overtone will be practically useful.

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16. Asymmetric biasing of the rf-SET in the case of significant $C_x$ leads to asymmetric effective capacitances, $C_1 \neq C_2$. The numerical results show that such asymmetry provides a slightly better performance of the rf-SET (which eliminates the concern about asymmetric biasing raised in Ref. 5).
17. Experimentally, the MR mode is preferable if the next stage noise is large, while if it is small, the OS mode is preferable. At low temperatures, $V_n$ in the OS mode is significantly smaller than in the MR mode.
20. After submission of the manuscript, the proposed mode of resonant $n$th overtone has been experimentally realized by the group of Keith Schwab. The rf-SET sensitivity for $n=2$ and $n=3$ has been found to be practically the same as in the conventional regime $n=1$.  

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**FIG. 3.** Dependence of (a) and (b) rf-SET response and (c) and (d) sensitivity on temperature $T$ in the MR and OS modes for several $Q$-factors. $V_0 = 0$, $R_x R_0 = 2000$, and $\omega = \omega_0$.

**FIG. 4.** (a) rf-SET response and (b) sensitivity in the regimes when the second or third overtone of the incident rf wave is in resonance with the tank circuit. $T = 0.01 e^2/C_S$, $R_x / R_0 = 2000$.  

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**TABLE 1.** Properties of the rf-SET response and sensitivity in the MR and OS modes.

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**FIG. 4.** (a) rf-SET response and (b) sensitivity in the regimes when the second or third overtone of the incident rf wave is in resonance with the tank circuit. $T = 0.01 e^2/C_S$, $R_x / R_0 = 2000$.  

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**TABLE 2.** Properties of the rf-SET response and sensitivity in the MR and OS modes.

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