Inter-Session Network Coding with Strategic Users: A Game-Theoretic Analysis of the Butterfly Network

Hamed Mohsenian-Rad, Member, IEEE Jianwei Huang, Senior Member, IEEE Vincent W. S. Wong, Senior Member, IEEE Sidharth Jaggi, Member, IEEE and Robert Schober, Fellow, IEEE

Abstract—We analyze inter-session network coding in a wired network using game theory. We assume that users are selfish and act as strategic players to maximize their own utility, which leads to a resource allocation game among users. In particular, we study a butterfly network, where a bottleneck link is shared by network coding and routing flows. We assume that network coding is performed using pairwise XOR operations. We prove the existence of Nash equilibrium for a wide range of utility functions. We also show that the number of Nash equilibria can be large (even infinite) for certain choices of parameters. This is in sharp contrast to a similar game setting with traditional packet forwarding, where the Nash equilibrium is always unique. We characterize the worst-case efficiency bound, i.e., the Price-of-Anarchy (PoA), compared to an optimal and cooperative network design. We show that by using a discriminatory pricing scheme which charges encoded and forwarded packets differently, we can improve the PoA in comparison with the case when a single pricing scheme is used. However, even when a discriminatory pricing scheme is used, the PoA is still worse than for the case when network coding is not applied. This implies that, although inter-session network coding can improve performance compared to routing, it is much more sensitive to users’ strategic behavior.

Keywords: Inter-session network coding, butterfly network, game theory, Nash equilibrium, price-of-anarchy, efficiency bound.

I. INTRODUCTION

Network coding is performed by jointly encoding multiple packets either from the same user or from different users. The former is intra-session network coding [1] while the latter is inter-session network coding [2], [3]. A common assumption in most prior network coding literature is that users are cooperative and do not pursue their own interests. However, this assumption can be violated in practice. Therefore, assuming that the users are selfish and strategic, in this paper we ask the following key questions: (a) What is the impact of users’ strategic behavior on network performance? (b) How does this impact change with different link pricing schemes?

It is widely accepted that pricing can improve the efficiency of resource allocation in distributed settings. In [4], Kelly et al. showed that if users are price-taker (i.e., they treat network prices as fixed), efficient resource allocation is achieved by properly setting congestion prices on each shared link. Recently, Johari et al. studied how the results can change in capacity-constrained [5] and capacity-unconstrained [6] networks if users are price-anticipator who realize that the price is impacted by each user’s behavior. In this case, users play a game, and the efficiency of resource allocation is characterized by the Nash equilibrium. A key performance metric is the Price-of-Anarchy (PoA), which measures the worst-case efficiency loss at a Nash equilibrium due to users’ price anticipating behavior. The PoA equals 1 if there is no efficiency loss. A smaller PoA indicates more efficiency loss.

The game theoretic analysis of network coding has received limited attention, e.g., in [7]–[13]. The results in [7]–[10] focus on intra-session network coding. In [12], the authors calculated the PoA for a class of inter-session network coding games that use reverse carpooling. Their analysis is specific to wireless networks while our focus is on wired networks. Moreover, in [12], users’ strategies are their choices of unicast routes. Here, users’ strategies are rather defined as their data rates. Users can also decide on whether and at what rate they want to participate in network coding. Since we take into account the links’ cost functions and the users’ utility functions, the PoA is evaluated with respect to the optimal solution of a network surplus maximization problem. The authors in [13] considered a game theoretic analysis of inter-session network coding between two users that share a link. It is shown that a rate allocation mechanism can enforce cooperation among users. In this paper, we assume that there are $N \geq 2$ users in a wired network, two of which can perform network coding via pairwise XOR operations, while the rest only use routing. This setting helps us better understand the interaction between network coding and routing flows. Moreover, we consider the impact of the utility functions of users, the cost of side links, price anticipation, price discrimination, and the PoA which are all not addressed in [13]. Our contributions are as follows:

- New problem formulation: We formulate the problem of maximizing the network surplus for inter-session network coding. This problem has not been studied before.
- Innovative pricing: We introduce a two discriminatory pricing scheme that charges network coding and routing packets differently. This new pricing is a better choice in reflecting the actual load generated by each user.
- Characterization of Nash equilibria: We prove that a Nash equilibrium always exists but it may not be unique.
- PoA calculation with zero-cost side links: Even with the new pricing method, the PoA is still smaller (i.e., worse) than the case without network coding. In fact, the PoA can be as low as 25%, which is less than the well-known 67% worst-case efficiency in [6] for routing networks.
- PoA Calculation with non-zero-cost side links: We show that if the side links in the butterfly network have non-zero cost, then the PoA can further reduce to only 20%, where no user is willing to participate in network coding.

The key results of this paper together with a comparison with the related state-of-the-art results for the case without

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network coding in [6] are summarized in Table I.

### II. Background: Resource Allocation

#### Game with Routing Flows

We first review a resource allocation game described in [4]–[6], where multiple end-to-end users compete to send packets through a shared link as in Fig. 1. No inter-session network coding is performed in this case. We will briefly summarize the results in [6], which serves as benchmark for our later discussions. In Fig. 1, a set of users $N = \{1, \ldots, N\}$ shares the bottleneck link $(i, j)$ between nodes $i$ and $j$. All packets that arrive at node $i$ are simply forwarded to node $j$ through link $(i, j)$. For each user $n \in N$, we denote the transmitter and receiver nodes by $s_n$ and $t_n$, respectively. Let $x_n$ denote the transmission rate of user $n \in N$. We assume that each user $n \in N$ has a utility function $U_n$, representing its satisfaction about its data rate $x_n$. On the other hand, the shared link has a cost function $C$, which depends on the total rate (i.e., $\sum_{n \in N} x_n$). As in [6], we make the following assumptions:

**Assumption 1:** For each $n \in N$, $U_n(x_n)$ is concave, nonnegative, increasing, and differentiable.

**Assumption 2:** The cost and price functions for link $(i,j)$ are chosen such that $C(q) = \int_0^q p(z) \, dz$. In particular, we assume that the link cost function is quadratic, $C(q) = \frac{1}{2} q^2$, and the link price function is linear, $p(q) = aq$. Quadratic cost functions and linear price functions are the only cost and price functions that satisfy the four axioms of rescaling, consistency, additivity, and positivity in cost-sharing systems [14].

Assumption 1 is used to model applications with elastic traffic, e.g., file transfer protocol (FTP) [4]. Examples of utility functions that satisfy Assumption 1 include the $\alpha$-fair utility functions with $\alpha \in (0, 1)$ [15]. Assumption 2 is also common in the network resource management (cf. [16]). In practice, cost function $C$ may reflect the actual cost of transmitting units of data over link $(i,j)$ or simply an approximate of the delay that the packets experience over link $(i,j)$. The more the aggregate data on the link, the higher is the average delay.

Let $\mathbf{x} = (x_1, \ldots, x_N)$. Given complete knowledge and centralized control of the network, an efficient rate allocation can be reached as a solution of the following problem:

**Problem 1:**

\[
\begin{aligned}
\text{maximize} & \quad \sum_{n=1}^N U_n(x_n) - C\left(\sum_{n=1}^N x_n\right) \\
\text{subject to} & \quad x_n \geq 0, \quad n = 1, \ldots, N.
\end{aligned}
\]

The objective function in Problem 1 is the network aggregate surplus [16, 17]. Problem 1 is a convex program. Therefore, if link $(i,j)$, or another network authority, has full control over the end-users, then optimal resource management can be achieved by forcing users to set their rates according to the centrally obtained optimal solutions of Problem 1. However, in practice, users may have full control over their own transmission rates. As a result, a distributed approach is more desirable.

To implement a distributed resource management, link $(i,j)$ can use pricing. In particular, following the price-based design in [4], link $(i,j)$ may introduce a single price:

\[
\mu(x) = p \left(\sum_{n=1}^N x_n\right)
\]

for each unit of data rate it carries. Each user $n \in N$ then pays $x_n \mu(x)$ for its data rate $x_n$ that goes into the shared link.

Next, we analyze how the users set their rates based on the price set by link $(i,j)$. If users are price takers, then each user $n \in N$ selects its rate $x_n$ to maximize its own surplus (utility minus payment) by solving the following local problem [4]:

\[
\max_{x_n \geq 0} \quad U_n(x_n) - x_n \mu(x) \quad \Rightarrow \quad x_n = U_n^{-1}(\mu).
\]

From the first fundamental theorem of welfare economics, if each user $n \in N$ selects its rate as in (2), then the network aggregate surplus is maximized at equilibrium [17, p. 326].

Next, we consider price anticipating users, where each user anticipates the effect of its data rate on the price. In this case, each user $n \in N$ no longer selects its rate as in (2). Instead, it strategically selects $x_n$ to maximize its surplus given the knowledge that the price $\mu(x)$ is set according to (1) and is not fixed; rather it depends on user $n$'s strategy $x_n$, as well as all other users' strategies $x_{-n}$. Clearly, the decision made by user $n$ also depends on the rates selected by other users, leading to a resource allocation game among all users:

**Game 1:**

- **Players:** Users in set $N$.
- **Strategies:** Transmission rates $x$ for all users.

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the cost models and other users’ utility functions.

In Game 1, each user \( n \in N \) selects its rate \( x_n \geq 0 \) to maximize its payoff \( P_n(x_n; x_{-n}) \). At a Nash equilibrium \( x^* = (x^*_1, \ldots, x^*_N) \), no user \( n \in N \) can increase its payoff by unilaterally changing its strategy \( x_n \). We note that, Game 1, as well as all other games that we define in this paper are games with complete information, where users are aware of the cost models and other users’ utility functions.

**Definition 1:** Let \( x^S = (x^S_1, \ldots, x^S_N) \) be an optimal solution for Problem 1 and \( x^* \) be a Nash equilibrium for Game 1 for the same choice of system parameters. We can define:

\[
\text{Efficiency at } x^* = \frac{\sum_{n=1}^N U_n(x^*_n) - C \left( \sum_{n=1}^N x^*_n \right)}{\sum_{n=1}^N U_n(x^S_n) - C \left( \sum_{n=1}^N x^S_n \right)}.
\]

**Definition 2:** The price-of-anarchy is the worst-case efficiency of a Nash equilibrium of Game 1 among all possible choices of system parameters (i.e., number of users, utility, cost, and price functions) under Assumptions 1 and 2.

The following key result is based on [6, Theorem 3]:

**Theorem 1:** Game 1 has a unique Nash equilibrium and

\[
\text{PoA (Game 1, Problem 1)} = \frac{2}{3} \approx 67\%.
\]

The PoA indicates how bad the network performance can become due to strategic behavior of end-users.

III. INTER-SESSION NETWORK CODING GAMES WITH ZERO SIDE LINK COSTS

In this section, we reformulate Problem 1 and Game 1 for a network with both routing and inter-session network coding flows. We show that the new game may have multiple Nash equilibria and the PoA will significantly reduced to 25%.

A. Problem Formulation

Consider the butterfly network in Fig. 2 [18]. Compared to Fig. 1, it has two direct side links \((s_1, t_N)\) and \((s_N, t_1)\). The source node of user 1 is located closer to the destination node of user \( N \) than to its own destination node (and vice versa). Thus, users 1 and \( N \) can perform inter-session network coding. In this regard, we must distinguish two types of users:

- **Network Coding Users:** Users 1 and \( N \), who can perform inter-session network coding.
- **Routing Users:** Users \( 2, \ldots, N-1 \), who cannot perform inter-session network coding.

Let \( X_1 \) and \( X_N \) denote packets sent from source nodes \( s_1 \) and \( s_N \), respectively. Node \( i \) can jointly encode packets \( X_{1} \) and \( X_N \) using pairwise XOR operations, and then send out the resulting encoded packet, denoted by \( X_{1} \oplus X_N \), towards node \( j \) (and from there towards \( t_1 \) and \( t_N \)). Given the remedy data \( X_1 \) from the side link \((s_1, t_N)\) and the remedy data \( X_N \) from the side link \((s_N, t_1)\), nodes \( t_N \) and \( t_1 \) can again use XOR operation to decode the encoded packets that they receive. In fact, nodes \( t_1 \) and \( t_N \) can decode both \( X_{1} \) and \( X_N \). The benefit of network coding is to reduce the load on link \((i, j)\) (thus reducing the cost) while achieving the same data rates compared to the case that no network coding is performed.

**Assumption 3:** Side links \((s_1, t_N)\) and \((s_N, t_1)\) in Fig. 2 have zero cost and impose zero prices.

For example, if the link cost is used to model the link delay and the side links \((s_1, t_N)\) and \((s_N, t_1)\) have a higher capacity than the shared link \((i, j)\), then the costs of the side links can be neglected. The case where the side links have non-zero cost is studied in Section IV. For the network in Fig. 2, the network aggregate surplus maximization problem becomes:

**Problem 2:**

\[
\begin{align*}
\text{maximize} & \quad \sum_{n=1}^N U_n(x_n) - C \left( \sum_{n=2}^{N-1} x_n + \max(x_1, x_N) \right) \\
\text{subject to} & \quad x_n \geq 0, \quad n = 1, \ldots, N.
\end{align*}
\]

The intuition behind the objective in Problem 2 is as follows. Since \( x_1 \) and \( x_N \) are selected independently by users 1 and \( N \), in general, \( x_1 \neq x_N \). Thus, regardless of the choice of an efficient network coding scheme, node \( i \) can network code only at rate \( \min(x_1, x_N) \). Those packets which are not encoded (e.g., at rate \( x_1 - \min(x_1, x_N) \) if \( x_1 \geq x_N \), and at rate \( x_N - \min(x_1, x_N) \) if \( x_1 \leq x_N \)) are simply forwarded, leading to a total rate \( \sum_{n=2}^{N-1} x_n + \max(x_1, x_N) \) on link \((i, j)\). If \( x_1 = x_N \), then all packets from users 1 and \( N \) are jointly encoded.

**Theorem 2:** Let \( x^S = (x^S_1, \ldots, x^S_N) \) be an optimal solution for Problem 2. We have \( x^S_1 = x^S_N \).

The proof is based on solving the Karush-Kuhn-Tucker (KKT) optimality conditions. Problem 2 can be solved in a distributed fashion again via pricing. Following the same pricing scheme in Section II, the shared link may apply a single price for all (i.e., coded and routed) packets:

\[
\mu(x) = p \left( \sum_{n=2}^{N-1} x_n + \max(x_1, x_N) \right).
\]

Each user \( n \) pays \( x_n \mu(x) \). However, this causes double charging for encoded packets. Thus, the single pricing model in (5)
leads to more payment from users than the actual link cost. This can be avoided by price discrimination, i.e., charging the routed and network-coded packets with different prices.

Let $\mu(x)$ in (5) denote the price to be charged for routed packets. Under the discriminatory pricing scheme, we define another price $\delta(x)$ for network coded packets. We have

$$\delta(x) = \beta \mu(x),$$

where $0 < \beta \leq 1$. Since encoded packets carry data from users 1 and $N$, they are both charged for the delivery of an encoded packet. As a result, if $\beta > \frac{1}{2}$, then the combined payment from users 1 and $N$ for delivery of an encoded packet becomes higher than the payment that each user makes for the delivery of a routed packet. Similarly, if $\beta < \frac{1}{2}$, then the combined payment from users 1 and $N$ for delivery of an encoded packet becomes lower than the payment that each user makes for the delivery of a routed packet. Therefore, in this paper, we focus on the case of $\beta = \frac{1}{2}$ because this is the only choice of $\beta$ that avoids over- or under-charging with two network coding flows. Based on this pricing scheme, user 1 pays $\min(x_1, x_N) \delta(x) + (x_1 - \min(x_1, x_N)) \mu(x)$. That is, it pays for transmission of its encoded packets at a price of $\delta(x)$ and for transmission of its forwarded (not coded) packets at a price of $\mu(x)$.

A similar payment model applies to user $N$. Each routing user $n = 2, \ldots, N - 1$ pays $x_n \mu(x)$.

We are now ready to define a resource allocation game for the network setting in Fig. 2, when users can anticipate prices $\mu$ and $\delta$ according to (5) and (6), respectively:

**Game 2:**
- **Players:** Users in set $N$.
- **Strategies:** Transmission rates $x$ for all users.
- **Payouts:** For network coding users 1 and $N$, we have

$$Q_1(x_1; x_{-1}) = \beta x_1 - (x_1 - (1 - \beta) \min(x_1, x_N)) \times p \left( \sum_{r=2}^{N-1} x_r + \max(x_1, x_N) \right),$$

$$Q_N(x_N; x_{-N}) = U_N(x_N) - x_N - (1 - \beta) \min(x_1, x_N) \times p \left( \sum_{r=2}^{N-1} x_r + \max(x_1, x_N) \right),$$

and each routing user $n \in N \setminus \{1, N\}$ has

$$Q_n(x_n; x_{-n}) = U_n(x_n) - x_n p \left( \sum_{r=2}^{N-1} x_r + \max(x_1, x_N) \right).$$

In the rest of this section, we answer the following questions:

1) Does Game 2 always (i.e., for any choice of system parameters) have a Nash equilibrium?
2) If a Nash equilibrium exists for Game 2, is it unique?
3) What is the worst-case efficiency (i.e., the PoA) at a Nash equilibrium of Game 2?

**B. Existence and Non-uniqueness of Nash Equilibria**

A Nash equilibrium of Game 2 is a non-negative data rate vector such that for all users $n \in N$, we have $Q_n(x_n; x_{-n}) \geq Q_n(x_n^*; x_{-n}^*)$ for any choice of $x_{-n} \geq 0$.

**Theorem 3:** Game 2 has at least one Nash equilibrium.

The proof of Theorem 3 is the direct application of the Rosen’s existence theorem for N-person games [19, Theorem 1] and is omitted. It is based on showing that for all users $n \in N$, the payoff function $Q_n(x_n; x_{-n})$ is a concave function with respect to $x_n$, even though $Q_1$ and $Q_N$ are not differentiable due to the max and min functions. Regarding the second question in Section III-A, we will see in Section III-C that Game 2 may have multiple Nash equilibria.

**C. Users’ Best Responses**

The strategic behavior of users can be modeled based on their best responses. In this regard, each user $n \in N$ selects its data rate as $x_n^B$ to maximize its own payoff $Q_n$, given $x_{-n}$:

$$x_n^B(x_{-n}) = \arg \max_{x_n \geq 0} Q_n(x_n; x_{-n}), \quad \forall n \in N.$$  (8)

Since problem (8) is concave, for each routing user $n \in N \setminus \{1, N\}$, $x_n^B(x_{-n})$ is the solution of

$$U_n(x_n) - a \left( \sum_{r=2, r\neq n}^{N-1} x_r + \max(x_1, x_N) \right) - 2ax_n = 0. \quad (9)$$

However, the best response for network coding users 1 and $N$ is more complex, due to the non-differentiability of the payoff functions $Q_1(x_1; x_{-1})$ and $Q_N(x_N; x_{-N})$. In fact, network coding user 1 should separately examine two scenarios:

(a) Selecting its strategy $x_1$ to be greater than or equal to $x_N$:

$$\hat{x}_1^B(x_{-1}) = \arg \max_{0 \leq x_1 \leq x_N} U_1(x_1) - (x_1 - (1 - \beta)x_N) \times a \left( \sum_{n=2}^{N-1} x_n + x_1 \right).$$

(b) Selecting its strategy $x_1$ to be less than or equal to $x_N$:

$$\hat{x}_1^B(x_{-1}) = \arg \max_{0 \leq x_1 \leq x_N} U_1(x_1) - \beta x_1 a \left( \sum_{n=2}^{N-1} x_n + x_1 \right).$$

In (10), since $x_1 \geq x_N$, we have: $\min(x_1, x_N) = x_N$ and $\max(x_1, x_N) = x_1$. Thus, $Q_1(x_1; x_{-1})$ reduces to the objective function in (10). In (11), since $x_1 \leq x_N$, we have: $\min(x_1, x_N) = x_1$, $\max(x_1, x_N) = x_N$, and $x_1 - (1 - \beta) \min(x_1, x_N) = \beta x_1$. Thus, $Q_1(x_1; x_{-1})$ reduces to the objective function in (11). Given $\hat{x}_1^B(x_{-1})$ and $\hat{x}_1^B(x_{-1})$, if $Q_1(\hat{x}_1^B(x_{-1}); x_{-1}) \geq Q_1(\hat{x}_1^B(x_{-1}); x_{-1})$, then user 1 selects its best response $\hat{x}_1^B(x_{-1}) = \hat{x}_1^B(x_{-1})$; otherwise, it selects $x_1^B(x_{-1}) = \hat{x}_1^B(x_{-1})$. The best response for user $N$ is obtained similarly. For user 1, the data rate $\hat{x}_1^B(x_{-1})$ is obtained as the value of $x_1 \geq x_N$ that satisfies

$$U_1'(x_1) - a \left( \sum_{n=2}^{N-1} x_n + x_1 \right) + a(1 - \beta)x_N - ax_1 = 0. \quad (12)$$

If $U_1(x_1)$ is non-linear, then $\hat{x}_1^B(x_{-1})$ is obtained as the value of $x_1 \in [0, x_N]$ that satisfies

$$U_1'(x_1) - \beta a \left( \sum_{n=2}^{N-1} x_n + x_1 \right) = 0. \quad (13)$$

When the utility function $U_1(x_1)$ is linear (i.e., $U_1(x_1)$ is a constant for all $x_1 \geq 0$), we have $\hat{x}_1^B(x_{-1}) = x_N$, if $U_1'(x_1) > \beta a(\sum_{n=2}^{N-1} x_n + x_1)$; and $\hat{x}_1^B(x_{-1}) = 0$, if $U_1'(x_1) < \beta a(\sum_{n=2}^{N-1} x_n + x_1)$. If $U_1'(x_1) = \beta a(\sum_{n=2}^{N-1} x_n + x_1)$, then $\hat{x}_1^B(x_{-1})$ can be any value between 0 and $x_N$. 
D. Nash Equilibrium and Price-of-Anarchy

Let $X^*$ denote the set of all Nash equilibria of Game 2. Recall that set $X^*$ has at least one member as shown in Theorem 3. By definition, for any Nash equilibrium $x^* \in X^*$, given $x_{n}^*$, we have $x_{n}^* = x_{n}^*$ for all $n \in \mathcal{N}$. Thus, all Nash equilibria of Game 2 can be obtained using (11), (12), (13) that only depend on the first derivatives of the utility functions. Therefore, for each Nash equilibrium $x^* \in X^*$, if we define the following linear utility functions:

$$
U_n(x_n) = U'_n(x^*_n) \
\forall n \in \mathcal{N},
$$

then $x^*$ continues to be a Nash equilibrium for a new game with new utilities $U_1(x_1), \ldots, U_N(x_N)$ in fact, $x^*$ is a Nash equilibrium for the family of games with utility functions $U_1(x_1), \ldots, U_N(x_N)$ having their first derivatives equal to $U'_1(x^*_1), \ldots, U'_N(x^*_N)$ at Nash equilibrium, respectively [6].

**Theorem 4:** Let $\sigma = \max \{U'_2(x^*_2), \ldots, U'_{N-1}(x^*_{N-1}), U'_1(x^*_1) + U'_N(x^*_N)\}$. For each Nash equilibrium $x^* \in X^*$ of Game 2 and any optimal solution $x^0$ of Problem 2, we have:

$$
\sum_{n=1}^{N} U_n(x_n^*) - C\left(\sum_{n=1}^{N-1} x_n^* + \max(x_1^*, x_N^*)\right) \\
\sum_{n=1}^{N} \sum_{n=1}^{N} U_n(x_n^*) - C\left(\sum_{n=1}^{N-1} x_n^* + \max(x_1^*, x_N^*)\right) \\
\geq \max_{q \in \mathbb{R}^+} \left\{ \sigma \bar{q} - C(\bar{q}) \right\}.
$$

The proof of Theorem 4 is similar [6, Lemma 4]. Note that, $\max_{q \in \mathbb{R}^+} \left\{ \sigma \bar{q} - C(\bar{q}) \right\}$ is the optimal objective of Problem 2 when utilities are linear. Thus, the right hand side in (15) is the efficiency for linear utility functions while the left hand side is the efficiency for any utility function, assuming that other parameters are fixed. We can rewrite Theorem 4 as:

**Theorem 5:** The worst-case efficiency at a Nash equilibrium of Game 2 occurs when the utility functions are linear. That is, $U_n(x_n) = \gamma_n x_n$, where $\gamma_n > 0$ for all users $n \in \mathcal{N}$.

From Theorem 5, the efficiency at Nash equilibrium depends on the concavity (i.e., the second derivative) of the utility functions. Note, that a linear utility is a least concave utility function that satisfies Assumption 1. Next, we obtain the value(s) of the Nash equilibrium(s) and PoA for Game 2.

**Theorem 6:** Suppose the utility functions are linear. Assume that $N \geq 2$ and let $x^*$ denote the Nash equilibrium for Game 2, without loss of generality, assume that $\gamma_1 \geq \gamma_N$. For notational simplicity, we define $q^* = \sum_{n=2}^{N} x_n^*$. (a) If $\gamma_N \leq \gamma_1 \leq \frac{1}{1+\beta} \gamma_N - \beta a q^*$, then

$$
\begin{align*}
0, \quad & \frac{\gamma_1 - \beta a q^*}{\alpha(1+\beta)} \\
\leq x_1^* = x_N^* & \leq \max \left\{ 0, \frac{\gamma_N - \beta a q^*}{\beta a} \right\}.
\end{align*}
$$

(b) If $\frac{1}{1+\beta} \gamma_N - \beta a q^* \leq \gamma_1 \leq \frac{2}{\beta} \gamma_N - \beta a q^*$, then

$$
x_1^* = \frac{\gamma_N}{\beta a} - q^*, \quad x_N^* = \frac{\gamma_N - \gamma_1}{\alpha(1+\beta)} - \frac{q^*}{1-\beta}.
$$

(c) If $\gamma_1 \geq \frac{2}{\beta} \gamma_N - \beta a q^*$, then

$$
x_1^* = \max \left\{ 0, \frac{\gamma_1}{2a} - q^* \right\}, \quad x_N^* = 0.
$$

(d) For any choice of system parameters in (a)-(c), each routing user $n = 2, \ldots, N - 14$ has the following rate

$$
x_n^* = \begin{cases} 
0, & \text{if } \gamma_n \leq a(q^* + x_1^*), \\
\frac{\gamma_n}{a^2} - q^* - x_1^*, & \text{otherwise}.
\end{cases}
$$

The proof of Theorem 6 is given in Appendix A. From Theorem 6(a), if the slopes of the linear utility functions for users 1 and $N$ (i.e., $\gamma_1$ and $\gamma_N$) are identical or close, then users 1 and $N$ choose equal rates and there is an infinite number of Nash equilibria. Theorem 6(b) and 6(c) show that if $\gamma_1$ and $\gamma_N$ are not close, then users 1 and $N$ choose different rates at the Nash equilibrium. Comparing this with the results in Theorem 2, we shall expect a drastic efficiency loss, especially if $\gamma_1 \geq \frac{2}{\beta} \gamma_N - \beta a q^*$ as it results in $x_N^* = 0$.

To study the properties of Nash equilibrium of Game 2, we consider two different cases:

**1) Two Users Case:** Assume that $N = 2$. In this case, the butterfly network includes two network coding users and no routing user. We can obtain the Nash equilibrium using Theorem 6 by setting $q^* = 0$ and show the following:

**Theorem 7:** In a network as in Fig. 2 with $N = 2$, under the single pricing scheme ($\beta = 1$),

$$
\text{PoA (Game 2, Problem 2)} = \frac{1}{3};
$$

and under the discriminatory pricing scheme with $\beta = \frac{1}{2}$,

$$
\text{PoA (Game 2, Problem 2)} = \frac{12}{25}.
$$

The proof of Theorem 7 is given in Appendix B. For this simple two-user scenario, inter-session network coding with no price discrimination can reduce the PoA from 0.67 in Theorem 1 to 0.33. Even if we use price discrimination by setting $\beta = \frac{1}{2}$, i.e., users 1 and $N$ split the price of encoded packets, the PoA improves only to $\frac{12}{25} = 0.48$. This implies that inter-session network coding is very sensitive to strategic users.

Note that, these results do not imply superiority of routing over network coding. For example, we can numerically verify that at any Nash equilibrium of Game 2, the surplus is no less than the surplus at the Nash equilibrium of Game 1 for the same choices of system parameters. That is, the absolute performance of non-cooperative network coding is no worse than the absolute performance of non-cooperative routing. However, the relative performance in non-cooperative network coding compared to optimal cooperative network coding is worse than the relative performance in routing case.

Numerical results on efficiency of the Nash equilibrium of Game 2 for 200 randomly generated scenarios with different choices of system parameters in the two-user case are shown in Fig. 3. In particular, in each scenario, the utility functions of the users are chosen to be $\sigma$-fair (cf. [15]) with a randomly selected utility parameter $\alpha \in (0, 1)$. We can see that by using price discrimination with parameter $\beta = \frac{1}{2}$, the guaranteed worst-case efficiency bound (i.e., the PoA) improves from 0.33 to 0.48. For the rest of this paper, we focus on the case with $\beta = \frac{1}{2}$. That is, the network coding users split the charge of transmitting their jointly encoded packets.
2) General Case: Next, consider the case where $N > 2$ users in the network. The presence of both network coding and routing users makes the analysis more complex. To see this, consider the network in Fig. 2 and assume that $N=3$, $a=1$, $\beta = \frac{1}{2}$, $\gamma_1 \geq \gamma_3$, $\gamma_3 = 1$, and $\gamma_2 = 3$. In this case, users 1 and 3 are the network coding users and user 2 is a routing user. From Theorem 6, the Nash equilibria are obtained as shown in Fig. 4. We can numerically verify that in this scenario, the worst-case efficiency at Nash equilibrium of Game 2 is 46.5%. Comparing this with the results in Theorem 7, we can expect that adding routing users will further reduce the PoA. This is shown in the next theorem for a general case:

**Theorem 8:** Assume that $N \geq 2$. (a) If the price discrimination parameter $\beta = \frac{1}{2}$, we have

$$\text{PoA (Game 2, Problem 2)} = \frac{1}{4}$$

(b) The worst-case efficiency occurs when $N \to \infty$.

The proof of Theorem 8 is given in Appendix C. Comparing Theorems 1, 7, and 8 we can see that a resource allocation game with both network coding and routing users has a worse PoA than the routing only and network coding only cases.

IV. Inter-Session Network Coding Games with Non-Zero Side Link Costs

In this section, we study the case where the side links have non-zero cost and show that the network coding users are no longer interested in participating in network coding in this case. This can further reduce the PoA to only 20%.

A. Problem Formulation

Consider the network in Fig. 5. In this figure, the side link $(s_1, t_N)$ has price $p_1$ and cost $C_1$ while the side link $(s_N, t_1)$ has price $p_N$ and cost $C_N$. Suppose that Assumption 2 also holds for the price and cost functions of both side links. In addition, we make the following assumption.

**Assumption 4 (Non-Zero Cost for Side Links):** The side links $(s_1, t_N)$ and $(s_N, t_1)$ in Fig. 5 always have non-zero cost and impose non-zero prices. In particular, the side link $(s_1, t_N)$ has price $p_1 = a_1 v_1$ for $a_1 > 0$ and the side link $(s_N, t_1)$ has price $p_N = a_N v_N$ for $a_N > 0$.

Clearly, by sending remedy packets over side link $(s_1, t_N)$, user 1 is helping user $N$ to decode the encoded packets it may receive. However, due to non-zero cost at the side links, user 1 will be charged for sending these remedy packets. A similar statement is true for user $N$. Therefore, users 1 and $N$ may decide to reduce the rate at which they send the remedy packets. Users 1 and $N$ can inform node $i$ about their decision via packet marking. Let $y_1$ and $z_1$ denote the rate at which source $s_1$ sends data to node $i$ marked for routing and network coding. Data rates $y_N$ and $z_N$ are defined for user $N$ similarly. Node $i$ may encode only those packets which are marked for...
network coding, at rate \( \min(z_1, z_N) \). Node \( i \) simply forwards the rest of packets\(^1\), at rate \( \sum_{n=1}^{N} y_n + |z_1 - z_N| \). Therefore, the total rate on link \((i, j)\) becomes \( \sum_{n=1}^{N} y_n + \max(z_1, z_N) \).

At destination node \( t_1 \), a packet coming from node \( i \) that is marked for network coding is collected and assumed to carry useful information only if it is accompanied by a remedy packet from node \( s_N \); otherwise, such packet is dropped. Similarly, at destination node \( t_N \), a packet coming from node \( i \) that is marked for network coding is collected only if it is accompanied by a remedy packet from node \( s_1 \); otherwise, such packet is dropped. Finally, we denote \( v_1 \) and \( v_N \) as the rates at which sources \( 1 \) and \( N \) send packets on side links \((s_1, t_N)\) and \((s_N, t_1)\). The routing users \( 2, \ldots, N-1 \) send routing packets at rates \( y_2, \ldots, y_{N-1} \). Let \( y = (y_1, \ldots, y_N) \), \( z = (z_1, z_N) \), and \( v = (v_1, v_N) \). For the network in Fig. 5, the network aggregate surplus maximization problem becomes

**Problem 3:**

\[
\begin{align*}
\text{maximize} & \sum_{n=2}^{N-1} U_n(y_n) + U_1(y_1 + \min(z_1, v_N)) \\
& + U_N(y_N + \min(z_N, v_1)) \\
& - C\left(\sum_{n=1}^{N} y_n + \max(z_1, z_N)\right) - C_1(v_1) - C_N(v_N) \\
\text{subject to} & \quad y_n \geq 0, \quad n = 1, \ldots, N, \quad z_1, z_N, v_1, v_N \geq 0.
\end{align*}
\]

Following a discriminatory pricing model as in Section III-A, we can define a resource allocation game for the network setting in Fig. 5, when users are price anticipators:

**Game 3:**

- **Players:** Users in set \( N \).
- **Strategies:** Transmission rates \( y, z, \) and \( v \).
- **Payoffs:** \( W_n(.) \) for each user \( n \in N \), where

\[
W_1(y_1, z_1, v_1; y_{-1}, z_N, v_N) = U_1(y_1 + \min(z_1, v_N)) - v_1p_1(v_1) - (y_1 + z_1 - (1 - \beta)\min(z_1, z_N))
\]

\[
p\left(\sum_{n=1}^{N} y_r + \max(z_1, z_N)\right),
\]

\[
W_N(y_N, z_N, v_N; y_{-N}, z_1, v_1) = U_N(y_N + \min(z_N, v_1)) - v_Np_N(v_N) - (y_N + z_N - (1 - \beta)\min(z_1, z_N))
\]

\[
p\left(\sum_{n=1}^{N} y_r + \max(z_1, z_N)\right),
\]

and for each \( n \in N \setminus \{1, N\} \), we have

\[
W_n(y_n; y_{-n}) = U_n(y_n) - y_n
\]

\[
\times p\left(\sum_{r=1}^{N} y_r + \max(z_1, z_N)\right).
\]

Here, \( y_{-n} = (y_1, \ldots, y_{n-1}, y_{n+1}, \ldots, y_N) \).

**B. Users’ Best Responses**

For network coding user \( 1 \), the best response is denoted by \( (y^B_1(y_{-1}, z_N, v_N), z^B_1(y_{-1}, z_N, v_N), v^B_1(y_{-1}, z_N, v_N)) \), which is obtained as the solution of the following problem

\[
\left(y^B_1(y_{-1}, z_N, v_N), z^B_1(y_{-1}, z_N, v_N), v^B_1(y_{-1}, z_N, v_N)\right) = \arg\max_{y_1 \geq 0, z_1 \geq 0, v_1 \geq 0} W_1(y_1, z_1, v_1; y_{-1}, z_N, v_N).
\]

\(^1\)Alternatively, node \( i \) can unmark any packet that was marked for network coding by users \( 1 \) and \( N \) but it did not participate in network coding. However, we can show that this approach has no advantage over the scenario considered.

The best response for network coding user \( N \), denoted by \( (y^B_N(y_{-N}, z_1, v_1), z^B_N(y_{-N}, z_1, v_1), v^B_N(y_{-N}, z_1, v_1)) \) can be obtained similarly. Next, we can show the following.

**Proposition 1:** Users \( 1 \) and \( N \) always send zero remedy packets at the best responses of Game 3. That is, we always have \( v^B_1(y_{-1}, z_N, v_N) = 0 \) and \( v^B_N(y_{-N}, z_1, v_1) = 0 \).

Proposition 1 can be proved by noticing that the pay-off \( W_1(y_1, z_1, v_1; y_{-1}, z_N, v_N) \) is decreasing in \( v_1 \) and \( W_N(y_{-N}, z_1, v_1; y_{-N}, z_N, v_N) \) is decreasing in \( v_N \). Clearly, if the network coding users do not receive the remedy data from the side links, they cannot decode any encoded packet they may receive through the shared link \((i, j)\). In fact, we can further show the following.

**Proposition 2:** Users \( 1 \) and \( N \) always send zero network coding packets to node \( i \) as the best responses of Game 3. That is, \( z^B_1(y_{-1}, z_N, v_N) = 0 \) and \( z^B_N(y_{-N}, z_1, v_1) = 0 \).

Notice that if \( v_N = 0 \), then \( \min(z_1, v_N) = 0 \) and the payoff function for user 1 reduces to \( U_1(y_1) - v_1p_1(v_1) - (y_1 + z_1 - (1 - \beta)\min(z_1, z_N))p\left(\sum_{n=1}^{N} y_r + \max(z_1, z_N)\right) \). In that case, the payoff function is decreasing in \( z_1 \). A similar statement is true for network coding user \( N \).

**C. Nash Equilibrium and Price-of-Anarchy**

Given the results on the users’ best responses in Propositions 1 and 2, we can conclude that at any Nash equilibrium of Game 3, denoted by \( (y^*, z^*, v^*) \), we should indeed have

\[
z^*_1 = z^*_N = v^*_1 = v^*_N = 0.
\]

In other words, at a Nash equilibrium of Game 3, no users performs network coding. In that case, the Nash equilibria of Game 3 would be closely related to the Nash equilibria of Game 1. In fact, for any choice of parameters, if \( x^* \) is a Nash equilibrium of Game 1, then \( y^* = x^* \), \( z^* = 0 \), and \( v^* = 0 \) would be a Nash equilibrium of Game 3 for the same choice of system parameters. From this, together with the results in Theorem 1(a), we can conclude that Game 3 always has a unique Nash equilibrium. This leads to the following theorem.

**Theorem 9:** The worst-case efficiency of Game 3 occurs when the utility functions are linear.

The proof of Theorem 9 is similar to that of [6, Lemma 4]. From Theorem 9, to obtain the PoA for Game 3 for arbitrary utility functions (as long as they satisfy Assumption 1), it is sufficient to only analyze the case where all utility functions are linear. Furthermore, if the side links have a very large cost compared to the cost of the bottleneck link, the optimal performance is achieved with no network coding. In that case, the efficiency can be obtained by using Theorem 1 and the optimal network aggregate surplus for Problem 3 is the same as the optimal network aggregate surplus for Problem 1. In addition, the network aggregate surplus is the same at the Nash equilibrium of Game 3 and Game 1. However, for general choices of \( a_1 > 0 \) and \( a_N > 0 \), obtaining the PoA requires further investigation of the optimal solution of Problem 3.

**Theorem 10:** Consider the network coding system in Fig.
5 with \( N \geq 2 \) users. (a) We have

\[
\text{PoA} (\text{Game 3, Problem 3}) = \frac{1}{5}, \quad (24)
\]

(b) The worst-case efficiency occurs when \( N \to \infty \).

The proof of Theorem 10 is given in Appendix D. Comparing Theorem 10 and Theorem 8, we can see that a non-zero cost at the side links can further reduce the PoA in a network resource allocation game as the users do not perform network coding in this case. If the side link price parameters \( a_1 \) and \( a_N \) are significantly greater than the bottleneck link price parameter \( a \), then network coding is not an optimal solution and the efficiency loss follows from the results in Theorem 1. This is shown in Fig. 6. For the results in this figure, the network topology is assumed to be as in Fig. 5, where \( N \to \infty \), \( \gamma_1 = \gamma_N = 1 \), \( a = 1 \), and \( \gamma_n = \frac{1}{2} \) for all \( n \in \mathbb{N} \setminus \{1, N\} \). The side link price parameters \( a_1 = a_N \) vary from 0 (non-inclusive) to 10. If \( a_1 > 0 \) and \( a_N > 0 \) tend to zero, the efficiency becomes as low as \( \frac{2}{5} = 0.2 \) as Theorem 10 suggests. As \( a_1 = a_N \) increase and tend to infinity, Problem 3 becomes equivalent to Problem 1 (in terms of the optimal network aggregate surplus) and Game 3 becomes equivalent to Game 1 (in terms of network aggregate surplus at Nash equilibrium) which leads to an efficiency higher than \( \frac{2}{5} \approx 0.67 \) as Theorem 1 suggests (for the choice of parameters in Fig. 6, the efficiency approaches \( \frac{2}{5} = 0.4 \)). Numerical results on the efficiency of the Nash equilibrium of Game 3 for 200 random scenarios with different choices of system parameters in the two-user case are shown in Fig. 7. We can see that the simulations confirm Theorem 10.

V. More General Network Topologies

Although the butterfly networks in Figs. 2 and 5 are simple, they can be used as building blocks for more general networks. In fact, as shown in [20], [21], many networks can be modeled as superposition of multiple butterfly networks. As an example, consider the grid topology in Fig. 8 with nine nodes, 12 links, and six users. All links have non-zero cost and incur non-zero prices, as in Section IV. The pricing parameter for each link \( l \in \{1, \ldots, 12\} \) is denoted by \( a_l > 0 \). Users 1 and 2 can form a network coding pair over a butterfly network with shared links 6 and 7, side links 1 and 2 between \( s_1 \) and \( t_2 \), and side links 11 and 12 between \( s_2 \) and \( t_1 \). Node \( D \) can act as an intermediate node to encode packets from \( s_1 \) and \( s_2 \).

Similarly, users 3 and 4 can form a network coding pair over a butterfly network with shared links 4 and 9, side links 3 and 8 between \( s_3 \) and \( t_4 \), and side links 5 and 10 between \( s_4 \) and \( t_3 \). Node \( B \) can act as an intermediate node to encode packets from \( s_3 \) and \( s_4 \). Users 5 and 6 are routing users.

Following similar steps as in formulating Problem 3 and Game 3, we can formulate the network surplus maximization problem and the resource allocation game for the network in Fig. 8. Although it is difficult to obtain the PoA analytically, we can still estimate the PoA numerically. Note that, we only need to calculate \( y_n^* \) for \( n = 1, \ldots, 6 \), because we already know from Proposition 1 that at Nash equilibrium, all network coding rates are zero. This is done as follows. First, we randomly select an initial strategy \( y_n \) for each user \( n \). Then, users take random turns and each user \( n \) individually updates \( y_n \) given the most updated \( y_{-n} \) from other users. The successive calculation of the best response strategies will continue until no user changes its strategy, i.e., no user can improve its payoff.
by unilaterally changing its transmission rate. Since all the best response dynamics converged in the numerical examples that we considered, the users’ transmission rates at convergence are used as Nash equilibrium in our numerical results.

The numerical results are shown in Fig. 9. Here, we assume that the price parameters on the side links $a_1 = a_2 = a_3 = a_5 = a_6 = a_8 = a_{10} = a_{11} = a_{12}$ vary from 0 to 10. The price parameters on the inner links are $a_4 = a_7 = a_9 = a_4 = a_6 = a_8 = a_{10} = a_{11} = a_{12} = 1$. Utility functions are linear and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 3$ and $\gamma_5 = \gamma_6 = 1$. When the side links have non-zero costs, no user participates in network coding. We can see that efficiency can be as low as 0.22, when $a_1 = a_2 = a_3 = a_5 = a_8 = a_{10} = a_{12} \rightarrow \infty$, network coding is no longer an optimal solution and thus the efficiency at the Nash equilibrium increases, approaching the results in Theorem 1.

We must emphasize that the possibility for generalizing the results in this section is only a conjecture, as the observations are limited to the very specific example of the topology in Fig. 8. The special trend in Fig. 9 and its resemblance to the trend in Fig. 6 may not apply to different inter-session network coding topologies. Furthermore, we note that the results here are limited to the case when network coding in a grid topology is performed similarly as in a butterfly topology. However, when it comes to a large network, such as the grid network in Fig. 8, there can be other options to perform network coding, such as the construction of grid topologies [22].

VI. CONCLUSION

In this paper, we studied the impact of strategic network coding users on the efficiency of network resource allocation in a butterfly network, where a bottleneck link is shared by several users. Two of the users have the capability of performing inter-session network coding, and the rest perform routing. Even with this simple setup, the results are dramatically different from the routing-only case. In particular, there can be many (even infinite) Nash equilibria in the resulting resource allocation game. This is in sharp contrast to a similar game setting with traditional packet forwarding where the Nash equilibrium is always unique. Furthermore, the efficiency loss can be more severe than for the case without network coding. In a butterfly network when the side links have zero cost, the efficiency at Nash equilibrium can be as low as 25%. If the side links have non-zero cost, then the efficiency at Nash equilibrium can further reduce to only 20%. These results generalize the well-known result of guaranteed 67% worst-case efficiency for packet forwarding networks in [6].

APPENDIX

A. Proof of Theorem 6

Due to $\gamma_1 \geq \gamma_N$, we have $x_1^* \geq x_N^*$. We prove this by contradiction. Assume $x_1^* < x_N^*$. Since $U_1(x_1^*) = \gamma_1$, from (13), $x_1^* < x_N^*$ implies that $\gamma_1 \leq \beta a (q^* + x_N^*)$. Furthermore, since $U_N(x_N) = \gamma_N$, from (12), we have $\gamma_N = aq^* + 2ax_N^* - a(1-\beta)x_1^*$. Since $\gamma_N \leq \gamma_1$, it is required that $q^* + 2ax_N^* - a(1-\beta)x_1^* \leq \beta a (q^* + x_N^*) \Rightarrow q^*(1-\beta) + x_N^*(2-\beta) - (1-\beta)x_1^* \leq 0$. If $\beta = 1$, then the inequality in (25) reduces to $x_N^* \leq 0$ which contradicts the assumption that $0 \leq x_1^* < x_N^*$. On the other hand, if $0 < \beta < 1$, then we can further show that $q^*(1-\beta) + x_N^*(2-\beta) - (1-\beta)x_1^* \geq q^*(1-\beta) + (x_N^*-x_1^*)(1-\beta) > 0$, where the last inequality is because $x_1^* < x_N^*$. It is clear that (26) contradicts (25). Thus, for any $0 < \beta \leq 1$, $x_N^*$ cannot be greater than $x_1^*$ and we always have $x_N^* \leq x_1^*$.

Part (a): To show $x_1^* = x_N^*$, assume that $x_1^* \neq x_N^*$. Since $x_1^* \geq x_N^*$, then $x_1^* > x_N^*$. From (12), we have $x_1^* = \frac{\gamma_1 - aq^* + a(1-\beta)x_N^*}{2a} > x_N^* \Rightarrow \gamma_1 > (1+\beta)a x_N^* + aq^*$, and

$$\gamma_N \leq \beta a x_1^* + baq^*.$$ (28)

From (27) and (28), we also have

$$\gamma_N \leq \beta aq^* + \frac{\beta}{2} (\gamma_1 - aq^* + a(1-\beta)x_N^*) < \frac{\beta}{1+\beta} (\gamma_1 + baq^*).$$ (29)

Therefore, $(1+1/\beta)\gamma_N - \beta aq^* < \gamma_1$. However, this contradicts the assumption that $\gamma_1 \leq (1+1/\beta)\gamma_N - \beta aq^*$. Thus, we indeed have $x_1^* = x_N^*$. From this, together with (12), we have

$$\gamma_1 \leq \beta ax_1^* + aq^* + ax_1^* = aq^* + (1+\beta)ax_1^* \Rightarrow \frac{\gamma_1 - aq^*}{a(1+\beta)} \leq x_1^* = x_N^*,$$ (30)

and

$$\gamma_N \geq \beta aq^* + \beta ax_1^* \Rightarrow x_N^* = x_1^* \leq \frac{\gamma_N - \beta aq^*}{\beta a}.$$ (31)

Part (b): The condition in this scenario holds if and only if $(1+1/\beta)\gamma_N - \beta aq^* \leq \frac{2}{\beta} \gamma_N - aq^* \Rightarrow \gamma_N \geq \beta aq^*$. (32)

Moreover, since $\gamma_1 \leq \frac{2}{\beta} \gamma_N - aq^*$, we have $\frac{2}{\beta} \gamma_N - \gamma_1 - aq^* \geq 0$ and $x_N^*$ in (17) is non-negative. Since $x_1^* \geq x_N^*$, this also implies non-negativity of $x_1^*$. Next, we consider two cases:
Case I) Assume that $x_N^* > 0$. Similar to Part (a), we can show that $x_1^* > x_N^*$. From this, together with (12), we have $x_1^* = (\gamma_1 - aq^* + (1 - \beta) x_N^*)/(2a)$ and $\gamma_N = \beta a q^* + \beta a x_1^*$. The latter further results in $x_1^* = (\gamma_N - \beta a q^*)/(\beta a) = \gamma_N/(\beta a) - \beta a q^*$. Thus, we finally have:

$$
\frac{\gamma_N}{\beta a} - q^* = \frac{\gamma_1 - a q^* + (1 - \beta) x_N^*}{2 a}
$$

$$
x_1^* = \frac{\gamma_N}{\beta a} - \gamma_1 - a q^*.
$$

(33)

Case II) Assume that $x_N^* = 0$. In that case, from (12), we have $x_1^* = (\gamma_1 - a q^*)/(2a)$ and $\gamma_N \leq \beta a q^* + \beta a x_1^*$. Replacing the former in the latter, we have

$$
\gamma_N \leq \frac{\beta}{2} + \frac{\beta a q^*}{2} \quad \Rightarrow \quad \gamma_N \geq \frac{2}{\beta} \gamma_N - a q^*.
$$

(34)

From (34) and since $\gamma_1 \leq \frac{2}{\beta} \gamma_N - a q^*$, we have $\gamma_1 = \frac{2}{\beta} \gamma_N - a q^*$. From (12), $x_1^* = \gamma_N/(\beta a) - q^*$. (part (c): The proof is similar to Part (b). Two cases of $x_1^* > x_N^*$ and $x_1^* = x_N^*$ are considered.

Part (d): For each node $n \in N\{1, N\}$, at each Nash equilibrium $x^*$ of Game 2, we have $x_1^* = (\gamma_n - a(\gamma_n + x_1^*)) = x_1^*$. Thus, for linear utilities, the derivative of the objective function in (9) in $x_n$ is $\gamma_n - a(\gamma_n - x_1^*)$, which is always non-positive. Hence, the objective function is decreasing in $x_n$. Thus, $x_n^* = 0$. If $\gamma_n \geq a(\gamma_n + x_1^*)$, then the objective is convex, we have $x_n^* = \frac{\gamma_n}{\gamma_n + x_1^*}$. Together, these two cases result in (19).

B. Proof of Theorem 7

At optimality, we have $x_1^* = x_2^* = (\gamma_1 + \gamma_2)/a$. Thus, the optimal network surplus becomes

$$
\gamma_1 x_1^* + \gamma_2 x_2^* - \frac{a}{2} \left(\max\{x_1^*, x_2^*\}\right)^2 = \frac{(\gamma_1 + \gamma_2)^2}{2 a}.
$$

(35)

Next, we examine the efficiency for all the scenarios in Theorem 6(a), (b), (c), where $q^* = 0$.

Case I) If $\gamma_2 \leq \gamma_1 \leq (1 + 1/\beta)\gamma_2$, then the Nash equilibria are as in (16). Since there are multiple Nash equilibria, the worst-case efficiency for Game 2 is obtained by solving

$$
\min_{x_1^*} \left(\frac{(\gamma_1 + \gamma_N) x_1^* - \frac{a}{2} x_1^*}{\gamma_1 (1 + \beta)}\right) / \left(\frac{(\gamma_1 + \gamma_2)^2}{2 a}\right)
$$

subject to

$$
\gamma_1 (1 + \beta) a \leq x_1^* \leq \frac{2}{\beta} \frac{\gamma_1}{\beta a}.
$$

We can show that if $\beta = 1$, then the optimal objective value of problem (36) becomes $2/1 - 1/16 = 7/16 \approx 0.438$. On the other hand, if $\beta = \frac{1}{2}$, then the optimal objective value of problem (36) becomes $6/9 - 1/9 = 5/9 \approx 0.556$.

Case II) If $(1 + 1/\beta) \gamma_2 < \gamma_1 \leq (\frac{7}{\beta}) \gamma_2$ (note: this may hold only if $\beta < 1$), then $x_1^*$ and $x_N^*$ are as in (17) where $q^* = 0$.

The worst-case efficiency is obtained by solving

$$
\min_{x_1^*} \left(\frac{\gamma_2}{(\gamma_1 + \gamma_2)^2} \left(\frac{2 (1 - \beta) a}{\beta (1 - \beta)} \gamma_1 + \frac{5 \beta - 1}{\beta^2 (1 - \beta)} \gamma_2\right)\right)
$$

subject to

$$
(1 + 1/\beta) \gamma_2 < \gamma_1 \leq \frac{2}{\beta} \frac{\gamma_2}{\beta a}.
$$

By applying the KKT conditions, the optimal objective of the above optimization problem when $\beta = \frac{1}{2}$ becomes $\frac{12}{25} = 0.48$.

Case III) We assume that $\frac{7}{\beta} \gamma_2 < \gamma_1$. From Theorem 6(c), the Nash equilibrium is as in (18) and the worst-case efficiency is obtained by solving the following optimization problem

$$
\min_{\gamma_1, \gamma_2} \left(\frac{\gamma_1}{2 a} - \frac{a}{2} \left(\frac{\gamma_1}{2 a}\right)^2\right) / \left(\frac{(\gamma_1 + \gamma_2)^2}{2 a}\right)
$$

subject to

$$
0 \leq \frac{2}{\beta} \frac{\gamma_2}{\beta a} < \gamma_1.
$$

For $\beta = 1$, the optimal objective value becomes $\frac{1}{4} \approx 0.25$. For $\beta = \frac{1}{2}$, the optimal objective value becomes $\frac{12}{25} = 0.48$.

From Cases I and III, if $\beta = 1$, PoA (Game 2, Problem 2) = $\min \{\frac{7}{16}, \frac{1}{3}\} = \frac{1}{3}$. From Cases I, II, and III, if $\beta = \frac{1}{2}$, PoA (Game 2, Problem 2) = $\min \{\frac{5}{12}, \frac{12}{25}\} = \frac{12}{25}$.}

C. Proof of Theorem 8

The optimal surplus for linear utilities is $a^2/(2a)$. Case I) We assume that $\gamma_1 + \gamma_N = \sigma$. Similar to the proof of Theorem 7, here we obtain the PoA by examining all the scenarios in Theorem 6(a), (b), (c). First, assume that

$$
\gamma_1 \leq \gamma_1 \leq (1 + 1/\beta) \gamma_N - \beta a q^*,
$$

and $\gamma_1 \geq a q^*$. To obtain the worst-case efficiency for this scenario, we need to solve the following optimization problem:

$$
\min_{x_1^*, \gamma_1, \gamma_N, q^*} \frac{\sigma x_1^* + \sum_{n=2}^{N-1} x_n^* - \frac{a}{2} (q^* + x_1^*)^2}{a^2/(2a)}
$$

subject to

$$
\gamma_n = a(q^* + x_n^* + x_1^*), \quad x_n^* > 0, \quad n \neq 2, N-1,
$$

$$
\gamma_n \leq a(q^* + x_1^*), \quad x_n^* = 0, \quad n \neq 2, N-1,
$$

$$
\gamma_N - \beta a q^* + (1 + 1/\beta) \gamma_N - \beta a q^*.
$$

We can show that for any choice of $\beta$ the optimal objective value of the above optimization problem is $\frac{1}{4} = 0.25$. Next, assume that (38) holds and we have

$$
\gamma_1 \leq a q^*. \quad (39)
$$

We can show that in this scenario, the worst-case efficiency occurs if $N \to \infty$ and we have $x_1^* = x_N^* = 0$ and $aq^* = \frac{1}{2\beta} a q^*$. Thus, the worst-case efficiency when (38) and (39) hold is obtained as

$$
\frac{2}{\beta 3^2 + \beta 3 + 3^2} = 0.36.
$$

(40)

If $\beta = \frac{1}{2}$, then (40) becomes $\frac{1}{4} \approx 0.36$. Finally, we assume that

$$
(1 + 1/\beta) \gamma_1 - \beta a q^* \leq \gamma_1 \leq \frac{2}{\beta} \gamma_N - a q^*.
$$

We can show that the worst-case efficiency in this scenario is still as in (40).

Case II) We assume that $\gamma_1 + \gamma_N < \sigma$. Following similar steps as in Case I and also using [6, Theorem 3], the worst-case efficiency in this scenario becomes $\frac{1}{4} = 0.67$.\]
D. Proof of Theorem 10

Let $y^* = (y^*_1, \ldots, y^*_N)$, $z^* = (z^*_1, z^*_N)$, and $w^* = (w^*_1, w^*_N)$ be the solution for Problem 3. Define $\gamma_{\text{max}} = \max_{n \in N} \gamma_n$ and $M = \{ n : \gamma_n = \gamma_{\text{max}} \}$ with size $M = |M|$. We can verify that (a) If $\gamma_1 + \gamma_N \geq (1 + \frac{a_1 + a_N}{a}) \gamma_{\text{max}}$, then $z^*_1 = z^*_N = w^*_1 = w^*_N = 0$ and for each $n \in N$, we have $y^*_n = 0$. (b) If $\gamma_{\text{max}} \leq \gamma_1 + \gamma_N \leq (1 + (a_1 + a_N)/a) \gamma_{\text{max}}$, then $z^*_1 = z^*_N = v^*_1 = v^*_N = (\gamma_1 + \gamma_N)/(a + a_1 + a_N)$, (41)

and for each $n \in N$, we have $y^*_n = 0$. (c) If $\gamma_{\text{max}} \geq \gamma_1 + \gamma_N$, then $z^*_1 = z^*_N = v^*_1 = v^*_N = 0$ and for each $n \in N$, we have $y^*_n = \frac{a_{\gamma_n}}{a} \gamma_{\text{max}}$ while for each $n \in N \setminus M$, we have $y^*_n = 0$.

Next, from (23), for each user $n$, we have

$$y^*_n = \begin{cases} \gamma_{\text{max}} \gamma_n & \text{if } \gamma_n > 0 \sum_{r=1,r \neq n}^N y^*_r, \\ 0 & \text{if } \gamma_n \leq 0 \sum_{r=1,r \neq n}^N y^*_r. \end{cases}$$

Case I) If $\gamma_1 + \gamma_N \geq (1 + \frac{a_1 + a_N}{a}) \gamma_{\text{max}}$, then the maximum network surplus is $((\gamma_1 + \gamma_N)^2/(2(a_1 + a_N)))$. The worst-case efficiency is obtained by solving the following problem

$$\begin{align*}
\text{minimize} & \quad y^*, \gamma, a_1, a_N, N, \gamma_{\text{max}} \\
\text{subject to} & \quad (\sum_{n=1}^N \gamma_n y^*_n - \frac{a}{2} y^*_n^2) / ((\gamma_1 + \gamma_N)^2/(2(a_1 + a_N))) \\
\gamma_n \leq a q_n, & \quad y^*_n > 0, n \in N, \\
\gamma_n \geq a q_n, & \quad y^*_n < 0, n \in N, \\
\sum_{n=1}^N y^*_n = q > 0, & \quad \sum_{n=1}^N y^*_n = q = 0, n \in N, \\
\gamma_1 + \gamma_N \geq \gamma_{\text{max}}, & \quad n \in N, \\
y^*_n \geq 0, & \quad n \in N.
\end{align*}$$

We can show the optimal objective value is $\frac{a}{2} = 0.2$.

Case II) If $\gamma_{\text{max}} \leq \gamma_1 + \gamma_N \leq (1 + \frac{a_1 + a_N}{a}) \gamma_{\text{max}}$, then the worst-case efficiency is equal to or higher (i.e., better) than $\frac{a}{2}$. In particular, if $\gamma_{\text{max}} \geq \gamma_1 + \gamma_N$, then the worst-case efficiency is $\frac{\gamma_{\text{max}}}{a}$ which resembles the results in [6].

Combining the results above in Cases I and II, we have PoA (Game 3, Problem 3) = min $\{1, \frac{a}{2}, \frac{\gamma_{\text{max}}}{a}\} = \frac{a}{2}$.

REFERENCES


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