Gaussian Hybrid Digital/Analog Coding with Bandwidth Expansion and Side Information at the Decoder

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The communication scenario

- Gaussian sources and channels
- Bandwidth expansion factor: \( \kappa = \frac{m}{n} \geq 1 \)
- Separable problem:
  \( (D, P) \) achievable \( \iff R_{WZ}(D) \leq \kappa C(P) \)
Separation works but...

- **Thresholding effect:**
  - When channel is worse than expected, the bin index cannot be decoded.
  - When side info is worse than expected, the source codeword inside the decoded bin cannot be disambiguated.
  - In either case, the distortion jumps to $\sigma^2_Z$.

- **Leveling-off effect:**
  - When side info and/or channel is better than expected, the quality of the decoded source codeword does not improve.
  - For the case of better side info, the only improvement in distortion comes in the MMSE estimation stage.
Hybrid digital/analog coding

- Brings robustness to varying channel and/or side information quality

- Different contexts:
  - Side information or no side information
  - Bandwidth match or mismatch b/w source and channel

- Different techniques:
  - Superposition
  - Dirty paper coding
  - Power splitting
  - Bandwidth splitting
  - Embedding of analog information into digital codewords
Embedded analog information

- Kochman-Zamir (2009)

\[ T = U + kX \]
\[ U \perp X \]

\[ R > I(X; T) \] guarantees typicality of \( (X^N, U^N, T^N(i^*)) \)
• \((X^N, U^N, T^N(i^*), V^N, Y^N)\) guaranteed to be typical.
• \(R < I(T; Y, V)\) guarantees that no other \((T^N(i), V^N, Y^N)\) typical.
• MMSE estimate \(\hat{X}^N = aT^N(i^*) + bY^N + cV^N\)
Embedded analog information

• Any $R$ and $k$ can be chosen so long as

\[ I(X; T) < R < I(T; Y, V) \]

• Interestingly, the gap between the two rates closes and the MMSE estimate achieves the optimum distortion

\[ D_{opt} = \frac{\sigma_Z^2 \sigma_W^2}{P + \sigma_W^2} \]

when

\[ k^2 = \frac{P^2}{\sigma_Z^2(P + \sigma_W^2)} \]
Embedded analog information

- Also, after \( k \) is fixed for some target \((\sigma^2_{W_0}, \sigma^2_{Z_0})\), the system still achieves optimal distortion for varying \((\sigma^2_W, \sigma^2_Z)\) if

\[
\sigma^2_Z(P + \sigma^2_W) = \sigma^2_{Z_0}(P + \sigma^2_{W_0})
\]
Rest of this talk

- We generalize the Wilson-Narayananan-Caire scheme to cover bandwidth expansion.
  - Optimum distortion $D_{\text{opt}} = \frac{\sigma_Z^2}{\left(1 + \frac{P}{\sigma_W^2}\right)^\kappa}$ can be reached.

- We analyze its robustness for varying $(\sigma_W^2, \sigma_Z^2)$.

  - Same type of robustness is granted as in $\kappa = 1$.

  - We analyze its performance according to a proposed *min-max distortion loss* measure, and compare it to that of CDS and uncoded transmission, and to a lower bound.
The scheme for bandwidth expansion

\[ T^m = U^m + K_{m \times n}X^n \]

\[ U^m \perp X^n \]
The scheme for bandwidth expansion

- So long as $C_U$ with $\text{tr}(C_U) \leq mP$ and $K$ are selected such that

$$I(X^n; T^m) < I(T^m; Y^n, V^m)$$

$T^{mN}(i^*)$ can be decoded with the guarantee that

$$(X^{nN}, U^{mN}, T^{mN}(i^*), V^{mN}, Y^{nN})$$

is jointly typical.
The scheme for bandwidth expansion

• **Theorem**: With the choices

\[ C_U = P I \quad \text{and} \quad K = \begin{bmatrix} \lambda I_{n\times n} \\ 0_{(m-n)\times n} \end{bmatrix} \]

where

\[ \lambda = \frac{P \left[ (1 + \frac{P}{\sigma^2_W})^\kappa - 1 \right]}{\sigma_Z^2 (1 + \frac{P}{\sigma^2_W})} \]

the scheme achieves

\[ D = D_{\text{opt}} = \frac{\sigma_Z^2}{\left(1 + \frac{P}{\sigma^2_W}\right)^\kappa} \]
The scheme for bandwidth expansion

• **Corollary:** After $K$ is fixed for $(\sigma_{W_0}^2, \sigma_{Z_0}^2)$, the system still achieves optimal distortion for varying $(\sigma_W^2, \sigma_Z^2)$ if

$$\frac{(1 + \frac{P}{\sigma_W^2})^\kappa - 1}{\sigma_Z^2 (1 + \frac{P}{\sigma_Z^2})} = \frac{(1 + \frac{P}{\sigma_{W_0}^2})^\kappa - 1}{\sigma_{Z_0}^2 (1 + \frac{P}{\sigma_{W_0}^2})}$$
The scheme for bandwidth expansion

- As far as we know, there is only one more scheme out there which has similar robustness characteristics: The common description scheme (CDS) of Nayak-Tuncel-Gündüz (2010)
Common description scheme

- $R > I(X; S)$
- Search $i^*$ s.t. both $(Y^{nN}, S^{nN}(i^*))$ and $(V^{mN}, U^{mN}(i^*))$ are typical.
- $i^*$ uniquely decoded if $R < \kappa I(U; V) + I(Y; S)$
Distortion loss

- We define the distortion loss

\[ L(\sigma^2_W, \sigma^2_Z) = \frac{D(\sigma^2_W, \sigma^2_Z)}{D_{\text{opt}}(\sigma^2_W, \sigma^2_Z)} \]

a fixed scheme achieves at any \((\sigma^2_W, \sigma^2_Z)\).

- We also define

\[ L(\mathcal{R}) = \max_{(\sigma^2_W, \sigma^2_Z) \in \mathcal{R}} L(\sigma^2_W, \sigma^2_Z) \]

a fixed scheme achieves over the region \(\mathcal{R}\).

- Finally, \( \mathcal{L}(\mathcal{R}) = \min L(\mathcal{R}) \) where the minimization is over all schemes in a class.
Min-max distortion loss

• We computed
  - $\mathcal{L}_{\text{UNC}}(\mathcal{R})$
  - $\mathcal{L}_{\text{SEP}}(\mathcal{R})$
  - $\mathcal{L}_{\text{CDS}}(\mathcal{R})$
  - $\mathcal{L}_{\text{HDA}}(\mathcal{R})$

• **Theorem:** We always have
  \[\mathcal{L}_{\text{HDA}}(\mathcal{R}) < \mathcal{L}_{\text{CDS}}(\mathcal{R}) = \mathcal{L}_{\text{SEP}}(\mathcal{R})\]

• No clear winner between uncoded and HDA.
A lower bound

For any discrete $\mathcal{C} \subset \mathcal{R}$, we always have

$$\min\max_{(\sigma^2_W, \sigma^2_Z) \in \mathcal{R}} L(\sigma^2_W, \sigma^2_Z) \geq \min\max_{(\sigma^2_W, \sigma^2_Z) \in \mathcal{C}} L(\sigma^2_W, \sigma^2_Z)$$
But what does it mean?

• If we are interested in \( L(\sigma_w^2, \sigma_Z^2) \) only for a finite number of \((\sigma_w^2, \sigma_Z^2)\), we can pose it as a network problem:
But what does it mean?

- The problem is, we do not even know good outer bounds in general.
There is one exception where the problem reduces to the Gaussian Heegard-Berger problem, for which we know the distortion tradeoff.
Numerical comparison

$\sigma^2_Z$ fixed

$\sigma^2_W$ varying

- Uncoded is better
- HDA is better
- Lower bound

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Summary

- Generalized the Wilson-Narayanan-Caire scheme to bandwidth expansion.
- Defined the min-max distortion loss metric for robustness of any scheme against varying channel and/or side information quality.
- Showed that the generalized scheme is always better than CDS according to this metric.
- Introduced a lower bound to the min-max distortion loss, and compared one "computable" case to the performance of the generalized scheme as well as of the uncoded transmission.