Blind Multiuser Detection by Kurtosis Maximization/Minimization

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***Abstract*—In this letter, blind multiuser detection is studied in the presence of multipath distortion. The kurtosis of the receiver's output is minimized with respect to the receiver's parameters subject to multiple linear constraints. The constraints are dependent on the spreading codes of the user of interest. In order to optimally combine signals from multipath, the kurtosis is parameterized by the constraint vector as well and is further maximized. It is shown that under some conditions, the optimal constraint vector converges to the channel vector of the desired user irrespective of noise, and the proposed receiver ensures perfect cancelation of both intersymbol interference and multiuser interference. Meanwhile, a minimum mean-square-error receiver can be constructed from the constraint vector.**

***Index Terms*—Constrained optimization, kurtosis maximization, kurtosis minimization, multiuser detection.**

I. Introduction

RECENTLY, there exists significant interest in studying cumulant/kurtosis-based multiuser detection techniques due to the prevalence of code-division multiple-access (CDMA) technology. Different methods can be divided into two categories. One is to design a bank of detectors with each one detecting one user [1]. Thus, all users can be detected at the same time. This multiuser detection scheme can be implemented in the base station that is capable of processing large amount of data in parallel. The other one is to consider only the desired user [2]–[6], such as in the mobile station. With given spreading codes of the desired user, the detector is forced to satisfy a linear constraint such that signals from the user of interest are detected.

However, those algorithms to detect a user of interest have shown certain disadvantages. The algorithm in [3] exhibits local minima and the inability to optimally combine signal components from different paths. The approaches [4], [5] are batch-iterative algorithms based on a batch processing of a block of data. Their global convergence has not been established. Our previously developed constant modulus algorithm (CMA)-based method [6] requires proper initialization for the constraint vector by other methods such as the adaptive minimum output energy (MOE) algorithm [7].

In this letter, we propose a novel receiver design criterion that does not suffer from convergence problem, while successfully suppressing both intersymbol interference (ISI) and multiuser interference (MUI). It optimizes the kurtosis of the receiver output under some unknown constraints. The constraint vector can be jointly adapted with the receiver and converges to the channel vector irrespective of noise. This fact well motivates design of another less expensive linear equalizer—minimum mean-square-error (MMSE) equalizer—by treating the constraint vector as an estimate of the channel. Simulation examples are illustrated to validate our analysis.

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II. Data Model

Consider a J-user CDMA system. User j is assigned spreading codes $c_j(k)(k = 0, \ldots, P-1)$ to spread each symbol $w_j(n)$. After spreading, the chip sequence propagates through a linear multipath channel with a baseband impulse response $g_j(n)$ of order q. If $L = \nu P$ chip-rate samples from nP to $nP + L - 1$ are collected in a data vector $\boldsymbol{y}_n$ and the receiver is synchronized to the desired user (i.e., user 1), then $\boldsymbol{y}_n$ has the following form [7]

$$\boldsymbol{y}_n = \boldsymbol{C}_1 \boldsymbol{g}_1 w_1(n) + \boldsymbol{H}_{\text{int}} \boldsymbol{w}_{\text{int}}(n) + \boldsymbol{v}_n \tag{1}$$

where

$$\boldsymbol{C}_1 = \begin{bmatrix} c_1(0) & & 0 \\ \vdots & \ddots & c_1(0) \\ c_1(P-1) & & \vdots \\ 0 & \vdots & c_1(P-1) \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \quad \boldsymbol{g}_1 = \begin{bmatrix} g_1(0) \\ \vdots \\ g_1(q) \end{bmatrix}.$$

$\boldsymbol{w}_{\text{int}}(n)$ is an interference vector including both ISI and MUI, $\boldsymbol{H}_{\text{int}}$ is the corresponding signature matrix, $\boldsymbol{v}_n$ is an additive white Gaussian noise (AWGN) vector, and $\boldsymbol{g}_1$ is the channel vector of user 1. The signature waveform $\boldsymbol{C}_1\boldsymbol{g}_1$ of $w_1(n)$ is a convolution of spreading codes and channel coefficients. This particular structure will be exploited to derive a blind adaptive multiuser receiver that is capable of detecting $w_1(n)$ while suppressing both ISI and MUI.

III. Multiuser Detection by Constrained Kurtosis Optimization

Our goal is to design a kurtosis-based multiuser receiver $\boldsymbol{f}$ whose output z provides an estimate of $w_1(n)$

$$z = \boldsymbol{f}^H \boldsymbol{C}_1 \boldsymbol{g}_1 w_1(n) + \boldsymbol{f}^H \boldsymbol{H}_{\text{int}} \boldsymbol{w}_{\text{int}}(n) + \boldsymbol{f}^H \boldsymbol{v}_n. \tag{2}$$

Instead of maximizing kurtosis $|\text{CUM}_4(z)|$ (the fourth-order cumulant $\text{CUM}_4(z) = E\{|z|^4\} - 2E^2\{|z|^2\} - |E\{z^2\}|^2$) under output power constraint (or normalized by the power) [5], [8], our approach is to minimize it under multiple constraints

$$\min_{\boldsymbol{f}} \mathcal{J} = |\mathrm{CUM}_4(z)|, \quad \text{subject to } \boldsymbol{C}_1^H \boldsymbol{f} = \boldsymbol{g}. \tag{3}$$

Those constraints implicitly constrain the power of the desired symbol to be constant when (2) is considered. After deriving minimum stationary points of $\mathcal{J}$, it can be easily verified that detection of the current symbol from our desired user is ensured. At the minimum point, $\mathcal{J}$ becomes $\mathcal{J}_{min} = |\kappa_1||\boldsymbol{g}^H\boldsymbol{g}_1|^4$ for any preselected $\boldsymbol{g}$, where κ_1 is the kurtosis of the desired signal. The constraint vector $\boldsymbol{g}$ can be preselected. The desired user's power becomes $|\boldsymbol{g}^H\boldsymbol{g}_1|^2$, and interference is zeroed out. Clearly, the choice of $\boldsymbol{g}$ affects the receiver's performance. It is desirable to maximize the power of the desired symbol [9]. Therefore, we treat $\boldsymbol{g}$ as a parameterized vector and opt for maximizing $\mathcal{J}_{\min}$. Our criterion is then described as follows:

$$\max_{\boldsymbol{g}} \min_{\boldsymbol{f}} \mathcal{J} = |\mathrm{CUM}_4(z)|, \quad \text{subject to } \boldsymbol{C}_1^H \boldsymbol{f} = \boldsymbol{g}. \tag{4}$$

To seek its optimum, we construct a Lagrange function

$$\mathcal{J}_1 = |\mathrm{CUM}_4(z)| + \boldsymbol{\lambda}^H\left(\boldsymbol{C}_1^H\boldsymbol{f} - \boldsymbol{g}\right) + \left(\boldsymbol{f}^H\boldsymbol{C}_1 - \boldsymbol{g}^H\right)\boldsymbol{\lambda} \tag{5}$$

where $\boldsymbol{\lambda}$ is a vectorized Lagrange multiplier. In order to minimize $\mathcal{J}_1$ with respect to (w.r.t.) $\boldsymbol{f}$ and maximize it w.r.t. $\boldsymbol{g}$, we form our gradient-based recursions

$$\boldsymbol{f}_{n+1} = \boldsymbol{f}_n - \mu_f \nabla_{\boldsymbol{f}} \mathcal{J}_1 \tag{6}$$

$$\boldsymbol{g}_{n+1} = \boldsymbol{g}_n + \mu_g\left(\boldsymbol{I} - \frac{\boldsymbol{g}_n\boldsymbol{g}_n^H}{\boldsymbol{g}_n^H\boldsymbol{g}_n}\right)\nabla_{\boldsymbol{g}}\mathcal{J}_1 \tag{7}$$

where $\boldsymbol{I} - (\boldsymbol{g}_n\boldsymbol{g}_n^H/\boldsymbol{g}_n^H\boldsymbol{g}_n)$ suggests update only in the direction orthogonal to $\boldsymbol{g}_n$ [7]. We first solve the multiplier following the similar procedure as in the MOE approach [7]. Using (6) and (7), applying the constraint $\boldsymbol{C}_1^H\boldsymbol{f}_{n+1} = \boldsymbol{g}_n$ at each iteration, and noticing that $\nabla_{\boldsymbol{f}}\mathcal{J}_1 = \nabla_{\boldsymbol{f}}|\mathrm{CUM}_4(z)| + \boldsymbol{C}_1\boldsymbol{\lambda}$ and $\nabla_{\boldsymbol{g}}\mathcal{J}_1 = -\boldsymbol{\lambda}$, we have

$$\boldsymbol{\lambda} = \frac{1}{\mu_f}\left(\boldsymbol{C}_1^H\boldsymbol{C}_1\right)^{-1}\left(\boldsymbol{C}_1^H\boldsymbol{f}_n - \mu_f\boldsymbol{C}_1^H\nabla_{\boldsymbol{f}}|\mathrm{CUM}_4(z)| - \boldsymbol{g}_n\right) \tag{8}$$

where $\nabla_f|\mathrm{CUM}_4(z)|$ is the gradient w.r.t. $\boldsymbol{f}$

$$\begin{aligned}\nabla_{\boldsymbol{f}}|\mathrm{CUM}_4(z)| = &\,\mathrm{sign}\left(\mathrm{CUM}_4(z)\right) \\ &\cdot\left(2E\left\{|z|^2z^*\boldsymbol{y}\right\} - 4E\left\{|z|^2\right\}E\{z^*\boldsymbol{y}\}\right. \\ &\left. -4E\{z^2\}E\{z\boldsymbol{y}\}\right).\end{aligned} \tag{9}$$

Substituting (8) into (6) and (7), we finally obtain

$$\boldsymbol{f}_{n+1} = \boldsymbol{\Pi}_{\boldsymbol{c}}^{\perp}\left(\boldsymbol{f}_n - \mu_f\nabla_{\boldsymbol{f}}|\mathrm{CUM}_4(z)|\right) + \boldsymbol{C}_1\boldsymbol{A}\boldsymbol{g}_n \tag{10}$$

$$\begin{aligned}\boldsymbol{g}_{n+1} = &\,\boldsymbol{g}_n + \frac{\mu_g}{\mu_f}\left(\boldsymbol{I} - \frac{\boldsymbol{g}\boldsymbol{g}^H}{\boldsymbol{g}^H\boldsymbol{g}}\right) \\ &\cdot\boldsymbol{A}\left(\boldsymbol{g}_n - \boldsymbol{C}_1^H\boldsymbol{f}_n + \mu_f\boldsymbol{C}_1^H\nabla_{\boldsymbol{f}}|\mathrm{CUM}_4(z)|\right)\end{aligned} \tag{11}$$

where μ_f and μ_g are step sizes

$$\boldsymbol{A} = \left(\boldsymbol{C}_1^H\boldsymbol{C}_1\right)^{-1} \quad \boldsymbol{\Pi}_{\boldsymbol{c}}^{\perp} = \boldsymbol{I} - \boldsymbol{C}_1\boldsymbol{A}\boldsymbol{C}_1^H.$$

To obtain a unique maximum of $\mathcal{J}_{\min}$, $\boldsymbol{g}$ is normalized at each step. The gradient of the kurtosis involves multiplication of statistical averages. Therefore, we cannot directly apply instantaneous estimation commonly adopted in typical adaptive algorithms. A block of received data is then used to estimate the gradient. As suggested in [8], the ensemble averages are replaced by empirical averages that are then adaptively updated through the use of a forgetting factor $\alpha (0 < \alpha < 1)$.

The proposed approach requires iterations of both $\boldsymbol{f}$ and $\boldsymbol{g}$ to obtain the optimal solution. By contrast, a more straightforward unconstrained approach, which first decorrelates the received signal with the desired user's spreading codes $\boldsymbol{C}_1$ and then applies any conventional single-user blind equalizer [10], [11] to detect the desired user's symbol, only needs to update the receiver. However, the unconstrained method cannot guarantee global convergence to the desired user. According to [12], convergence to the desired user's symbol at a fixed delay in a mixed source environment is not guaranteed unless some signal-to-interference-and-noise ratio (SINR) and power conditions are satisfied. Since the decorrelated data clearly contain both ISI and MUI besides desired signal and noise, it is impossible to ensure the SINR and power conditions for any CDMA systems. Consequently, the convergence to the desired user's symbol is not ensured for the unconstrained method. By comparison, the proposed method exhibits global convergence as proved in the next section.

IV. Convergence Analysis

We now study the convergence property of our algorithm when interference and AWGN are present in the system. Assume kurtoses of input signals from all users have the same sign, i.e., all are either positive or negative. For convenience of analysis, assume they are the same and are denoted by κ. Denote the ith column of signature matrix $\boldsymbol{H} = [\boldsymbol{C}_1\boldsymbol{g}_1, \boldsymbol{H}_{\mathrm{int}}]$ by $\boldsymbol{h}_i$. Then $\boldsymbol{h}_1 = \boldsymbol{C}_1\boldsymbol{g}_1$. After considering a norm constraint on $\boldsymbol{g}$ and the zero kurtosis for the AWGN, (5) is transformed to

$$\begin{aligned}\mathcal{J}_1 = &\,|\kappa|\sum_i\left|\boldsymbol{f}^H\boldsymbol{h}_i\right|^4 + \boldsymbol{\lambda}^H\left(\boldsymbol{C}_1^H\boldsymbol{f} - \boldsymbol{g}\right) \\ &+ \left(\boldsymbol{f}^H\boldsymbol{C}_1 - \boldsymbol{g}^H\right)\boldsymbol{\lambda} + \rho(\boldsymbol{g}^H\boldsymbol{g} - 1).\end{aligned} \tag{12}$$

where scalar ρ is a multiplier. To obtain stationary points, we take the derivative w.r.t. both $\boldsymbol{f}$ and $\boldsymbol{g}$ and set them equal to zero. After expressing $|\boldsymbol{f}^H\boldsymbol{h}_i|^4$ as $(\boldsymbol{f}^H\boldsymbol{h}_i\boldsymbol{h}_i^H\boldsymbol{f})^2$, these derivatives then satisfy the following:

$$\begin{aligned}\nabla_{\boldsymbol{f}}\mathcal{J}_1 = &\,2|\kappa||\boldsymbol{f}^H\boldsymbol{h}_1|^2\left(\boldsymbol{h}_1^H\boldsymbol{f}\right)\boldsymbol{h}_1 \\ &+ 2|\kappa|\sum_{i\neq 1}|\boldsymbol{f}^H\boldsymbol{h}_i|^2\left(\boldsymbol{h}_i^H\boldsymbol{f}\right)\boldsymbol{h}_i + \boldsymbol{C}_1\boldsymbol{\lambda} = \boldsymbol{0}\end{aligned} \tag{13}$$

$$\nabla_{\boldsymbol{g}}\mathcal{J}_1 = -\boldsymbol{\lambda} + \rho\boldsymbol{g} = \boldsymbol{0}. \tag{14}$$

With $\boldsymbol{h}_1$ being replaced by $\boldsymbol{C}_1\boldsymbol{g}_1$, (13) becomes

$$[\boldsymbol{C}_1, \boldsymbol{H}_{\mathrm{int}}]\begin{bmatrix} 2|\kappa||\boldsymbol{f}^H\boldsymbol{C}_1\boldsymbol{g}_1|^2\left(\boldsymbol{g}_1^H\boldsymbol{C}_1^H\boldsymbol{f}\right)\boldsymbol{g}_1 + \boldsymbol{\lambda} \\ 2|\kappa||\boldsymbol{f}^H\boldsymbol{h}_2|^2\left(\boldsymbol{h}_2^H\boldsymbol{f}\right) \\ \vdots \end{bmatrix} = \boldsymbol{0}.$$

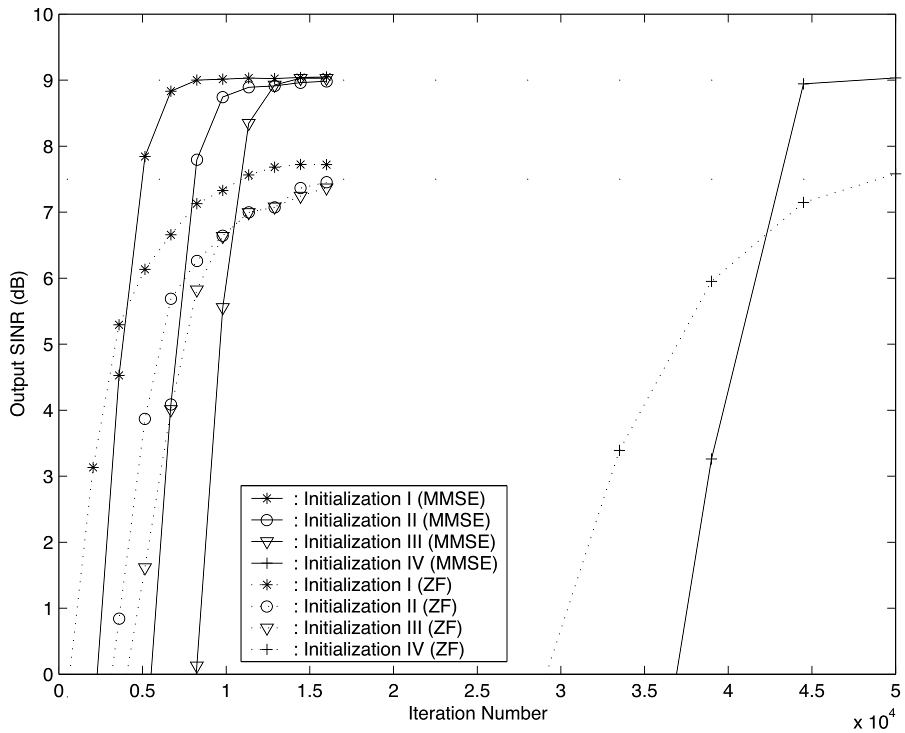

Fig. 1. Initialization effect.

Under an assumption that $[\boldsymbol{C}_1, \boldsymbol{H}_{\text{int}}]$ has full column rank and after applying constraint $\boldsymbol{C}_1^H \boldsymbol{f} = \boldsymbol{g}$, we immediately conclude that at stationary points

$$\boldsymbol{\lambda} = -2|\kappa||\boldsymbol{g}^H \boldsymbol{g}_1|^2 \left(\boldsymbol{g}_1^H \boldsymbol{g}\right) \boldsymbol{g}_1 \quad (15)$$

$$|\boldsymbol{f}^H \boldsymbol{h}_i| = 0 \text{ for } i \neq 1. \quad (16)$$

Considering (14) and (15), we have

$$\rho \boldsymbol{g} = -2|\kappa||\boldsymbol{g}^H \boldsymbol{g}_1|^2 \left(\boldsymbol{g}_1^H \boldsymbol{g}\right) \boldsymbol{g}_1.$$

This result shows that the optimal constraint vector is proportional to $\boldsymbol{g}_1$. Since $||\boldsymbol{g}|| = 1$, we obtain

$$\boldsymbol{g} = \frac{e^{j\theta}}{||\boldsymbol{g}_1||} \boldsymbol{g}_1 \quad \rho = -2|\kappa|||\boldsymbol{g}_1||^4 \quad (17)$$

where θ is an arbitrary phase. With (17), (15) becomes

$$\boldsymbol{\lambda} = -2|\kappa|e^{j\theta}||\boldsymbol{g}_1||^3 \boldsymbol{g}_1. \quad (18)$$

According to (16)–(18), we can observe at stationary points that: 1) the optimal receiver $\boldsymbol{f}$ is able to remove all ISI and MUI; 2) our optimal constraint vector converges to the normalized channel vector within a phase ambiguity; and 3) the multiplier $\boldsymbol{\lambda}$ is proportional to the channel vector. Next, we check properties of all stationary points.

According to (13), it is easily observed that the Hessian matrix of $\mathcal{J}_1$ w.r.t. $\boldsymbol{f}$ is zero at the stationary points, which does not provide much insight into the properties of those points. Therefore, we directly investigate the cost function in the neighborhood of stationary points

$$|\text{CUM}_4(z)| = |\kappa||\boldsymbol{g}^H \boldsymbol{g}_1|^4 + |\kappa| \sum_{i \neq 1} |\boldsymbol{f}^H \boldsymbol{h}_i|^4. \quad (19)$$

For any constraint $\boldsymbol{g}$, $|\text{CUM}_4(z)| \geq |\kappa||\boldsymbol{g}^H \boldsymbol{g}_1|^4$. If $\boldsymbol{f}^H \boldsymbol{h}_i \neq 0$ for some $i \neq 1$, then $|\text{CUM}_4(z)| > \mathcal{J}_{\min}$. Therefore, we conclude that $\boldsymbol{f}^H \boldsymbol{h}_i = 0$ for $i \neq 1$ is the global minimum point and correspondingly $\mathcal{J}_{\min} = |\kappa||\boldsymbol{g}^H \boldsymbol{g}_1|^4$. Considering the code constraint $\boldsymbol{C}_1^H \boldsymbol{f} = \boldsymbol{g}$ [(17) and (16)], the optimal $\boldsymbol{f}$ is then the solution of the following linear equation:

$$\boldsymbol{f}^H [\boldsymbol{C}_1, \boldsymbol{H}_{\text{int}}] = \left[\frac{e^{-j\theta}}{||\boldsymbol{g}_1||} \boldsymbol{g}_1^H, 0, \ldots, 0 \right].$$

It is thus a zero-forcing (ZF) equalizer. Therefore, detection of the desired symbol from the desired user is ensured. However, the ZF receiver may amplify the noise in detection, although the convergence is irrelevant to noise. Due to the convergence of the constraint vector to the channel vector, one might treat $\boldsymbol{g}$ as a channel estimate and construct an MMSE equalizer that can better deal with the noise. It will be compared with the proposed kurtosis-based equalizer in our simulation results.

V. Simulation

The performance of the proposed adaptive kurtosis-based ZF receiver and corresponding blind MMSE receiver constructed from updated constraint vector is studied. Comparisons with the adaptive MOE approach [7] and the unconstrained approach described in Section III are also included.

In the first simulation example, we examine the effect of initialization on the convergence of the proposed approaches in terms of SINR. We set $P = 8$, $J = 3$, $q = 3$, SNR $=$ 10 dB, $L = 16$, and $\alpha = 0.95$ in cumulant estimation [8]. The multipath channels are randomly generated and fixed. One hundred Monte Carlo runs are conducted to obtain the average SINR's for both kurtosis-based ZF and blind MMSE receivers. For clarity, only results corresponding to four random initialization settings of $\boldsymbol{f}$ and $\boldsymbol{g}$ are presented in Fig. 1. Those different initializations are found to yield amplitudes of the desired signal at the receiver's output as 0.719, 0.56, 0.43, and 0.0275

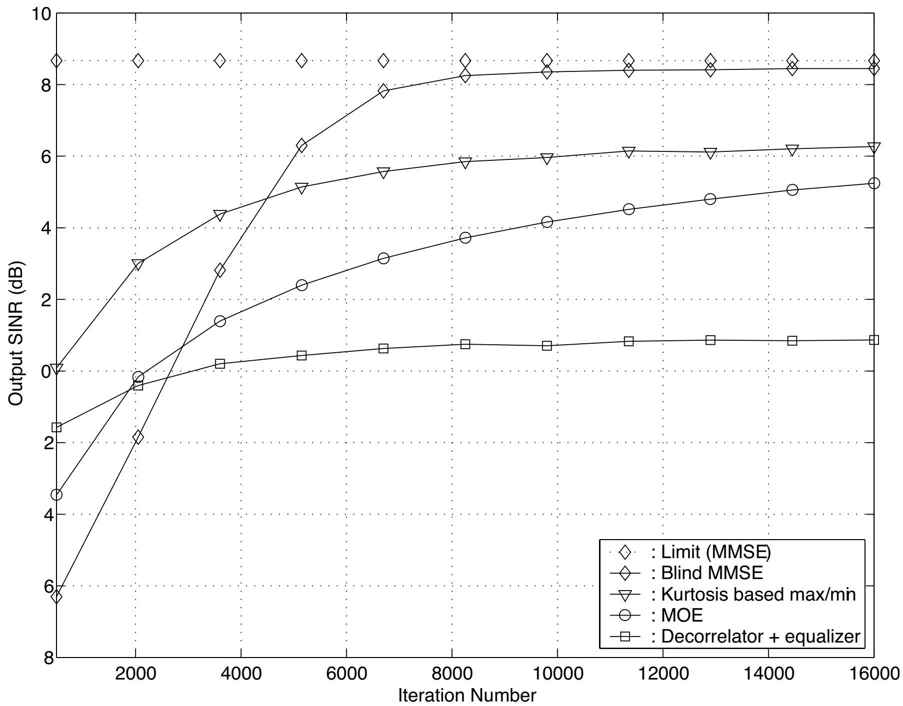

Fig. 2. Output SINR versus iteration number.

before iteration starts, and these are termed as Initializations I, II, III, and IV, respectively. It is observed that all the solid lines (MMSE) converge to a level of 9 dB and all the dotted lines (ZF) to a lower level of 7.5 dB, indicating that the convergence level is not affected by initialization. However, convergence rate highly depends on initialization as expected. The higher the initial power of the desired symbol, the faster the convergence. In an extreme setting where the desired signal power is close to zero (Initialization IV), the algorithm requires 5×10^4 iterations to converge.

In our second example, we take the same parameters as in the previous experiment for all approaches, except that channel coefficients for each user are randomly generated in each realization. Fig. 2 compares SINRs of different receivers versus symbol period (iteration number). For reference, we also show the performance of an ideal MMSE receiver constructed from true covariance matrix and channel parameters. It is seen that the proposed blind MMSE receiver performs best and approaches the ideal MMSE receiver after enough iterations. The proposed kurtosis-based receiver performs better than the MOE receiver [7]. The unconstrained approach shows the worst performance. Slow convergence of both proposed receivers stems from that in cumulant estimation, as observed in some existing methods [8].

VI. Conclusion

A blind multiuser detection technique to suppress both ISI and MUI is proposed. The method is based on the kurtosis max/min idea under multiple constraints on the receiver. The optimal receiver is able to cancel all interference even in an AWGN communication environment, and the optimal constraint vector is proportional to the channel vector.

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