PERFORMANCE ANALYSIS OF BLIND CHANNEL ESTIMATION FOR PRECODED MULTIRATE MULTIUSER SYSTEMS

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ABSTRACT

We have proposed a blind channel estimation method for a precoded multirate multiuser communication system. The method requires subspace decomposition of a data covariance matrix. In this paper we analyze the method in the presence of imperfect estimation of the covariance matrix due to finite data sample effect. Based on perturbation theory, the first order perturbation of the noise subspace is derived. Then the covariance and the mean square error of the channel estimates are obtained in closed forms. They explicitly depend on the statistical properties of the input and the additive noise up to the fourth order. Therefore, they can be evaluated for different scenarios.

1. INTRODUCTION

Recently, there have been increasing demands for multimedia services from mobile users in a multirate communication system. By employing precoding techniques before transmission, much flexibility in system design and optimization can be achieved [5], [6]. With such consideration, currently diverse structures can be unified in one seamless framework.

Under similar framework, we have proposed a blind channel estimation method for such a precoded multirate system with arbitrarily long channel impulse responses [9]. The method is based on a key observation that the various structured signature waveforms for the desired user are contributed by a common unknown channel. These signatures lie in the signal subspace, thus are orthogonal to the noise subspace. Therefore the total projections of signature waveforms onto the estimated noise subspace are minimized to obtain both the channel parameters and the delay of the user of interest.

In practice, the noise subspace is usually obtained from eigen decomposition of the sample covariance of the receiver data. Due to limited amount of received information, the estimation error is introduced to the covariance and finally results in an error in the proposed channel estimates. In this paper, we first derive an expression for the first order perturbation of the noise subspace. Then we derive the covariance and the mean square error (MSE) of the channel estimates in closed forms. Since these quantities turn out to be dependent on the statistical properties of the input and the additive noise up to the fourth order, we further simplify them based on our input/output data model. As an example, we provide closed form expressions when the input and the noise are real valued. These procedures can be followed for other scenarios.

2. PRECODED MULTIUSER COMMUNICATIONS AND CHANNEL ESTIMATION METHOD

Consider a multirate communication system with K users. Data rate of each user is adjusted by individual precoder employed in the transmitter. The input stream for user k is partitioned into M_k sub-streams and precoded by linear precoder $s_{k,m_k}(0), \cdots, s_{k,m_k}(P-1)$. Assume the communication channel has finite impulse responses $g_{k,l}(n)$ of order q_k. It can be factorized by P: $q_k = Q_k P + \eta_k$ where Q_k and η_k are both integers with $0 \leq \eta_k < P$. At the receiver, L antennas are used. Each time $\nu = QP$ samples are collected where ν satisfies $\nu \geq \max(P + Q_k P + P) = \max(Q_k + 2)P$. It has been shown that the received data vector has the following form [9]

$$\boldsymbol{y}(n) = \sum_{k=1}^{K} \sum_{m=1}^{M_k} \sum_{j=-(Q_k+1)}^{Q-1} \boldsymbol{h}_{k,m,j} b_{k,m}(n+j) + \boldsymbol{v}(n) \quad (1)$$

where $b_{k,m}(n+j)$ is the mth input of user k in the $(n+j)$th block, $\boldsymbol{h}_{k,m,j}$ is its signature waveform

$$\boldsymbol{h}_{k,m,j} = \boldsymbol{A}_{k,m,j}\boldsymbol{g}_k, \quad \boldsymbol{A}_{k,m,j} = \boldsymbol{I}_L \otimes (\boldsymbol{J}^{Pj+\tau_k}\boldsymbol{S}_{k,m}),$$

$$\boldsymbol{S}_{k,m} = \begin{bmatrix} s_{k,m}(0) & & 0 \\ \vdots & \ddots & s_{k,m}(0) \\ s_{k,m}(P-1) & & \vdots \\ 0 & \ddots & s_{k,m}(P-1) \\ \vdots & 0 & 0 \end{bmatrix},$$

τ_k is the delay in chip periods, $\boldsymbol{J}$ is a $\nu \times \nu$ shift matrix with all $1's$ in the first diagonal below the main diagonal, $\boldsymbol{g}_k$ is a channel vector, "$\otimes$" represents the Kronecker product, and $\boldsymbol{v}(n)$ is AWGN. After putting $b_{k,m}(n+j)$ for all j and m

at time n in a vector $\boldsymbol{b}_k(n)$, and similarly $\boldsymbol{h}_{k,m,j}$ in a matrix $\boldsymbol{H}_k$, (1) is simplified as

$$\boldsymbol{y}(n) = \sum_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{b}_k(n) + \boldsymbol{v}(n) = \boldsymbol{H}\boldsymbol{b}(n) + \boldsymbol{v}(n) \qquad (2)$$

where $\boldsymbol{H}$ is obtained by stacking $\boldsymbol{H}_k$ column by column, and $\boldsymbol{b}(n)$ is a long vector with $\boldsymbol{b}_k(n)$ as its entries.

In order to obtain the noise subspace, the data correlation matrix $\boldsymbol{R}$ is computed from (2) and decomposed

$$\boldsymbol{R} = E\{\boldsymbol{y}(n)\boldsymbol{y}^H(n)\} = \sum_{k=1}^{K} \gamma_k \boldsymbol{H}_k \boldsymbol{H}_k^H + \sigma_v^2 \boldsymbol{I}, \qquad (3)$$

$$\boldsymbol{R} = \boldsymbol{U}_s \boldsymbol{\Lambda}_s \boldsymbol{U}_s^H + \boldsymbol{U}_n \boldsymbol{\Lambda}_n \boldsymbol{U}_n^H, \qquad (4)$$

$$\boldsymbol{\Lambda}_s = diag(\lambda_i^2 + \sigma_v^2), \quad \boldsymbol{\Lambda}_n = \sigma_v^2 \boldsymbol{I}$$

where $\boldsymbol{U}_s$ spans the signal subspace which is also a range space of $\boldsymbol{H}$ and $\boldsymbol{U}_n$ spans the noise subspace. Then joint estimation of the delay and channel vector for the desired user - user 1 is formulated as follows

$$(\widehat{\tau_1}, \widehat{\boldsymbol{g}}_1) = \arg\min_{\tau, \boldsymbol{g}} \boldsymbol{g}^H [\sum_{m,j} \boldsymbol{A}_{1,m,j}^H \widehat{\boldsymbol{U}}_n \widehat{\boldsymbol{U}}_n^H \boldsymbol{A}_{1,m,j}] \boldsymbol{g} \qquad (5)$$

where m takes all integers from 1 to M_1, j takes all integers from $-(Q_1+1)$ to $Q-1$.

3. PERFORMANCE ANALYSIS

In practice, $\boldsymbol{R}$ is estimated from N data sample vectors $\widehat{\boldsymbol{R}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{y}(n)\boldsymbol{y}^H(n)$. The number of available data samples will affect the accuracy of the subspace estimate, thus the performance of the channel estimator. In this section, we first study the perturbation of the noise subspace. Then we investigate the covariance and MSE of our channel estimator as functions of N. The analyses will be based on the perturbation technique [3], [8]. However, we do not assume a particular distribution of the source [4] such as a Gaussian process assumed in most array signal processing work and used in obtaining MSE in our previous paper [9]. Since channel parameters are jointly estimated with time delay, the performance is also determined by the delay estimation error. This joint problem will be simplified by assuming perfect timing in the following derivation (for example, [1]).

3.1. Perturbation in the noise subspace

Our method employs the noise subspace of estimated correlation matrix instead of the data matrix as used in [1], [7]. Therefore the results therein are not directly applicable. However, similar procedures can be followed. We will denote a perturbation by preceding the corresponding quantity by δ. Then $\delta\boldsymbol{R} = \widehat{\boldsymbol{R}} - \boldsymbol{R}$. It is Hermitian and small when N is large enough [8]. This perturbation will cause an error in the estimate of noise subspace. For notational convenience, let $\boldsymbol{Z}$ be the noise-free data correlation matrix

$$\boldsymbol{Z} = \boldsymbol{R} - \sigma_v^2 \boldsymbol{I} = \boldsymbol{U}_s \boldsymbol{\Omega} \boldsymbol{U}_s^H, \quad \boldsymbol{\Omega} = diag\{\lambda_i^2\} \qquad (6)$$

and $\boldsymbol{X}$ be the objective matrix in the cost function without perturbation (see (5))

$$\boldsymbol{X} = \sum_{m,j} \boldsymbol{A}_{1,m,j}^H \boldsymbol{U}_n \boldsymbol{U}_n^H \boldsymbol{A}_{1,m,j}. \qquad (7)$$

Lemma 1: If $\boldsymbol{R}$ is perturbed to be $\boldsymbol{R} + \delta\boldsymbol{R}$, the first order perturbation of the noise subspace has the following form

$$\delta\boldsymbol{U}_n \approx -\boldsymbol{Z}^{\dagger} \delta\boldsymbol{R}\boldsymbol{U}_n \qquad (8)$$

where $(\cdot)^{\dagger}$ denotes pseudo-inverse.

Proof: According to [3], the perturbed noise subspace can be expressed by: $\widehat{\boldsymbol{U}}_n = \boldsymbol{U}_n + \boldsymbol{U}_s\boldsymbol{Q}$ where $\boldsymbol{Q}$ has a norm of the order of $||\delta\boldsymbol{R}||$. Since the perturbation scenario is different from that in [3], the expression for $\boldsymbol{Q}$ needs to be investigated. After perturbation, the subspace decomposition of $\widehat{\boldsymbol{R}}$ becomes (see (4))

$$\widehat{\boldsymbol{R}} = \widehat{\boldsymbol{U}}_s \widehat{\boldsymbol{\Lambda}}_s \widehat{\boldsymbol{U}}_s^H + \widehat{\boldsymbol{U}}_n \widehat{\boldsymbol{\Lambda}}_n \widehat{\boldsymbol{U}}_n^H.$$

Correspondingly, $\widehat{\boldsymbol{\Lambda}}_s$ and $\widehat{\boldsymbol{\Lambda}}_n$ are perturbed versions of $\boldsymbol{\Lambda}_s$ and $\boldsymbol{\Lambda}_n$. From the orthogonality, we have

$$(\boldsymbol{U}_n + \boldsymbol{U}_s\boldsymbol{Q})^H(\boldsymbol{R} + \delta\boldsymbol{R}) = (\boldsymbol{\Lambda}_n + \delta\boldsymbol{\Lambda}_n)(\boldsymbol{U}_n + \boldsymbol{U}_s\boldsymbol{Q})^H. \qquad (9)$$

Notice that $\boldsymbol{U}_n^H\boldsymbol{R} = \boldsymbol{\Lambda}_n\boldsymbol{U}_n^H$, $\boldsymbol{U}_s^H\boldsymbol{R} = \boldsymbol{\Lambda}_s\boldsymbol{U}_s^H$. After taking Hermitian on both sides of (9) and ignoring second order terms, we obtain

$$\delta\boldsymbol{R}\boldsymbol{U}_n + \boldsymbol{U}_s\boldsymbol{\Lambda}_s\boldsymbol{Q} = \boldsymbol{U}_s\boldsymbol{Q}\boldsymbol{\Lambda}_n + \boldsymbol{U}_n\delta\boldsymbol{\Lambda}_n \qquad (10)$$

where the Hermitian property of $\delta\boldsymbol{R}$ and $\delta\boldsymbol{\Lambda}_n$ has been used. Pre-multiplying both sides of (10) by $\boldsymbol{U}_s^H$, we have

$$\boldsymbol{U}_s^H\delta\boldsymbol{R}\boldsymbol{U}_n + \boldsymbol{\Lambda}_s\boldsymbol{Q} \approx \boldsymbol{Q}\boldsymbol{\Lambda}_n. \qquad (11)$$

Since $\boldsymbol{\Lambda}_n = \sigma_v^2\boldsymbol{I}$ and $\boldsymbol{\Omega} = \boldsymbol{\Lambda}_s - \sigma_v^2\boldsymbol{I}$, $\boldsymbol{Q}$ can be solved

$$\boldsymbol{Q} \approx -\boldsymbol{\Omega}^{-1}\boldsymbol{U}_s^H\delta\boldsymbol{R}\boldsymbol{U}_n.$$

Observing $\delta\boldsymbol{U}_n = \boldsymbol{U}_s\boldsymbol{Q}$ and (6), we obtain (8). □

Eq. (8) shows that $\delta\boldsymbol{U}_n$ is related to $\delta\boldsymbol{R}$. Notice that our solution has a form very similar to that in [3], although the corresponding terms have different meanings.

3.2. Channel estimation error

The bias of channel estimate pertaining to the proposed method is dependent on $\delta\boldsymbol{U}_n$ thus $\delta\boldsymbol{R}$. The following lemma provides an analytical result for channel estimate when $\boldsymbol{R}$ is not accurately estimated.

Lemma 2: Due to perturbation $\delta\boldsymbol{R}$, the first order perturbation of channel estimate is given by

$$\delta\boldsymbol{g}_1 \approx \sum_{m,j} \boldsymbol{W}_{m,j}\delta\boldsymbol{R}\boldsymbol{d}_{m,j} \qquad (12)$$

where $\boldsymbol{W}_{m,j}$ and $\boldsymbol{d}_{m,j}$ are deterministic quantities

$$\boldsymbol{W}_{m,j} \triangleq \boldsymbol{X}^{\dagger}\boldsymbol{A}_{1,m,j}^H\boldsymbol{U}_n\boldsymbol{U}_n^H, \quad \boldsymbol{d}_{m,j} \triangleq \boldsymbol{Z}^{\dagger}\boldsymbol{A}_{1,m,j}\boldsymbol{g}_1.$$

Proof: In the absence of subspace perturbation, $\boldsymbol{g}_1$ (which has been assumed to be unitary) is an eigenvector of the $(q_1+1)\times(q_1+1)$ matrix $\boldsymbol{X}$ associated to its unique null eigenvalue. According to (7), the perturbation of $\boldsymbol{X}$ due to $\delta\boldsymbol{U}_n$ has the form

$$\delta\boldsymbol{X} \approx \sum_{m,j} \boldsymbol{A}_{1,m,j}^H[\delta\boldsymbol{U}_n\boldsymbol{U}_n^H+\boldsymbol{U}_n\delta\boldsymbol{U}_n^H+\delta\boldsymbol{U}_n\delta\boldsymbol{U}_n^H]\boldsymbol{A}_{1,m,j}.$$

After substituting (8), $\delta\boldsymbol{X}$ is related to $\delta\boldsymbol{R}$ by

$$\delta\boldsymbol{X} \approx -\sum_{m,j} \boldsymbol{A}_{1,m,j}^H[\boldsymbol{Z}^\dagger\delta\boldsymbol{R}\boldsymbol{U}_n\boldsymbol{U}_n^H+\boldsymbol{U}_n\boldsymbol{U}_n^H\delta\boldsymbol{R}\boldsymbol{Z}^\dagger]\boldsymbol{A}_{1,m,j}. \tag{13}$$

Then $\delta\boldsymbol{g}_1$ has the following form [1]

$$\delta\boldsymbol{g}_1 \approx -\boldsymbol{X}^\dagger\delta\boldsymbol{X}\boldsymbol{g}_1. \tag{14}$$

After substituting (13) in (14) and noticing that $\boldsymbol{A}_{1,m,j}\boldsymbol{g}_1$ is orthogonal to $\boldsymbol{U}_n$, we obtain (12). □

According to (12), $\delta\boldsymbol{g}_1$ is a random vector due to the randomness of $\delta\boldsymbol{R}$. Its second order statistics can be evaluated based on the statistics of the received data. First its covariance has a form

$$E\{\delta\boldsymbol{g}_1\delta\boldsymbol{g}_1^H\} \approx \sum_{\substack{m_1,m_2\\ j_1,j_2}} \boldsymbol{W}_{m_1,j_1}\boldsymbol{B}_{m_1,m_2,j_1,j_2}\boldsymbol{W}_{m_2,j_2}^H \tag{15}$$

where deterministic quantities $\boldsymbol{B}_{m_1,m_2,j_1,j_2}$ are defined as

$$\boldsymbol{B}_{m_1,m_2,j_1,j_2} \triangleq E\{\delta\boldsymbol{R}\boldsymbol{d}_{m_1,j_1}\boldsymbol{d}_{m_2,j_2}^H\delta\boldsymbol{R}\}.$$

Since $\delta\boldsymbol{R}$ is related to the second order information of the received signal, each $\boldsymbol{B}_{m_1,m_2,j_1,j_2}$ thus depends on up to the fourth order statistics of the input and the noise. Its closed form expression is derived in *Appendix*, following similar procedures as used in a correlation matching context [10]. Once $\boldsymbol{B}_{m_1,m_2,j_1,j_2}$ is derived, the MSE of the estimated channel vector becomes

$$E\{\delta\boldsymbol{g}_1^H\delta\boldsymbol{g}_1\} \approx \sum_{\substack{m_1,m_2\\ j_1,j_2}} tr(\boldsymbol{W}_{m_1,j_1}\boldsymbol{B}_{m_1,m_2,j_1,j_2}\boldsymbol{W}_{m_2,j_2}^H) \tag{16}$$

with tr denoting the trace of a matrix. According to the expression of $\boldsymbol{B}_{m_1,m_2,j_1,j_2}$ in (26) in *Appendix*, it is clear that the MSE is proportional to $\frac{1}{N}$ which is similar to the results in [1]. Therefore as $N \to \infty$, the estimator will give exact channel information. This is not surprising since $\widehat{\boldsymbol{R}}$ will approach $\boldsymbol{R}$. The MSE will decrease if σ_v^2 decreases (or SNR increases). However, it is not obvious that the MSE decreases when the power of the transmitted signal increases although this holds for Gaussian sources [9].

Various simulations have been conducted to test our analytical results and the performance of the blind MMSE detector constructed from the estimated channel parameters. Unfortunately, we are unable to detail them due to lack of space. They will be reported elsewhere in the future. However, it has been observed that those results are highly consistent with our analytical results.

Appendix: Derivation of Matrix $\boldsymbol{B}_{m_1,m_2,j_1,j_2}$

For notational convenience, we temporarily denote $\boldsymbol{d}_{m_1,j_1}$ by $\boldsymbol{d}_1$, $\boldsymbol{d}_{m_2,j_2}$ by $\boldsymbol{d}_2$, and $\boldsymbol{B}_{m_1,m_2,j_1,j_2}$ by $\boldsymbol{B}_{1,2}$. Also denote $\boldsymbol{y}(n)$ in (2) by $\boldsymbol{y}_n$, $\boldsymbol{b}(n)$ by $\boldsymbol{b}_n$, and $\boldsymbol{v}(n)$ by $\boldsymbol{v}_n$. Substituting $\delta\boldsymbol{R}$ by $\widehat{\boldsymbol{R}}-\boldsymbol{R}$, we obtain

$$\boldsymbol{B}_{1,2} = E\{\widehat{\boldsymbol{R}}\boldsymbol{d}_1\boldsymbol{d}_2^H\widehat{\boldsymbol{R}}\} - \boldsymbol{R}\boldsymbol{d}_1\boldsymbol{d}_2^H\boldsymbol{R}.$$

Assume N data vectors are mutually independent for our derivation purpose. It can always be made possible by taking only those N vectors which are not contributed by common inputs, although data vectors consecutive in time may be dependent on each other due to channel span. Then

$$E\{\widehat{\boldsymbol{R}}\boldsymbol{d}_1\boldsymbol{d}_2^H\widehat{\boldsymbol{R}}\} = \frac{1}{N}\boldsymbol{C}_1 + (1-\frac{1}{N})\boldsymbol{R}\boldsymbol{d}_1\boldsymbol{d}_2^H\boldsymbol{R}$$

where $\boldsymbol{C}_1 \triangleq E\{\boldsymbol{y}_n\boldsymbol{y}_n^H\boldsymbol{d}_1\boldsymbol{d}_2^H\boldsymbol{y}_n\boldsymbol{y}_n^H\}$. Therefore

$$\boldsymbol{B}_{1,2} = \frac{1}{N}(\boldsymbol{C}_1 - \boldsymbol{R}\boldsymbol{d}_1\boldsymbol{d}_2^H\boldsymbol{R}). \tag{17}$$

According to (2), $\boldsymbol{y}_n\boldsymbol{y}_n^H$ can be expanded to four terms

$$\boldsymbol{y}_n\boldsymbol{y}_n^H = \boldsymbol{H}\boldsymbol{b}_n\boldsymbol{b}_n^H\boldsymbol{H}^H + \boldsymbol{H}\boldsymbol{b}_n\boldsymbol{v}_n^H + \boldsymbol{v}_n\boldsymbol{b}_n^H\boldsymbol{H}^H + \boldsymbol{v}_n\boldsymbol{v}_n^H.$$

Suppose the noise $\boldsymbol{v}_n$ is zero mean and white Gaussian. It is independent of the input sequence which is independent and identically distributed (i.i.d.) with zero mean, variance $\boldsymbol{\Gamma}$ and finite fourth-order moment. Since $E\{\boldsymbol{v}_n\boldsymbol{v}_n^H\} = \sigma_v^2\boldsymbol{I}$, $E\{\boldsymbol{b}_n\boldsymbol{b}_n^H\} = \boldsymbol{\Gamma}$. Then only following terms survive in $\boldsymbol{C}_1$

$$\begin{aligned} \boldsymbol{C}_1 &= \boldsymbol{H}\boldsymbol{C}_2\boldsymbol{H}^H + \boldsymbol{C}_3 \\ &+ \boldsymbol{H}E\{\boldsymbol{b}_n\boldsymbol{b}_n^T\}\boldsymbol{H}^T\boldsymbol{d}_2^*\boldsymbol{d}_1^T E\{\boldsymbol{v}_n^*\boldsymbol{v}_n^H\} \\ &+ E\{\boldsymbol{v}_n\boldsymbol{v}_n^T\}\boldsymbol{d}_2^*\boldsymbol{d}_1^T\boldsymbol{H}^* E\{\boldsymbol{b}_n^*\boldsymbol{b}_n^H\}\boldsymbol{H}^H \\ &+ \sigma_v^2(\boldsymbol{d}_2^H\boldsymbol{d}_1)\boldsymbol{H}\boldsymbol{\Gamma}\boldsymbol{H}^H + \sigma_v^2(\boldsymbol{d}_2^H\boldsymbol{H}\boldsymbol{\Gamma}\boldsymbol{H}^H\boldsymbol{d}_1)\boldsymbol{I} \\ &+ \sigma_v^2\boldsymbol{H}\boldsymbol{\Gamma}\boldsymbol{H}^H\boldsymbol{d}_1\boldsymbol{d}_2^H + \sigma_v^2\boldsymbol{d}_1\boldsymbol{d}_2^H\boldsymbol{H}\boldsymbol{\Gamma}\boldsymbol{H}^H \end{aligned} \tag{18}$$

where superscript $*$ denotes complex conjugate

$$\boldsymbol{C}_2 = E\{\boldsymbol{b}_n\boldsymbol{b}_n^H\boldsymbol{H}^H\boldsymbol{d}_1\boldsymbol{d}_2^H\boldsymbol{H}\boldsymbol{b}_n\boldsymbol{b}_n^H\},$$

$$\boldsymbol{C}_3 = E\{\boldsymbol{v}_n\boldsymbol{v}_n^H\boldsymbol{d}_1\boldsymbol{d}_2^H\boldsymbol{v}_n\boldsymbol{v}_n^H\}.$$

For a given system, the last form terms can be easily evaluated. However, the first four terms are dependent on the statistical properties of the transmitted signal and the noise. As an example, we restrict our attention to a scenario where all quantities real valued. For all other scenarios, one can follow the similar procedures detailed next. However, we will not enumerate them for our concise presentation.

Assume the variance of the input is σ_b^2 and the fourth order moment is m_{4b}. Then

$$E\{\boldsymbol{b}_n\boldsymbol{b}_n^T\} = \sigma_b^2\boldsymbol{I}, \quad E\{\boldsymbol{v}_n\boldsymbol{v}_n^T\} = \sigma_v^2\boldsymbol{I}. \tag{19}$$

To easily evaluate $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$, we perform "vec" operations [10] first to combine corresponding terms and then reverse operations "$unvec$" to obtain these matrices

$$\boldsymbol{C}_2 = unvec\,[\boldsymbol{C}_4 vec(\boldsymbol{H}^H \boldsymbol{d}_1 \boldsymbol{d}_2^H \boldsymbol{H})], \tag{20}$$

$$\boldsymbol{C}_3 = unvec\,[\boldsymbol{C}_5 vec(\boldsymbol{d}_1 \boldsymbol{d}_2^H)] \tag{21}$$

where the property of "vec" has been applied [2], and

$$\boldsymbol{C}_4 \triangleq E\{(\boldsymbol{b}_n \boldsymbol{b}_n^T) \otimes (\boldsymbol{b}_n \boldsymbol{b}_n^T)\}, \;\; \boldsymbol{C}_5 \triangleq E\{(\boldsymbol{v}_n \boldsymbol{v}_n^T) \otimes (\boldsymbol{v}_n \boldsymbol{v}_n^T)\}$$

Then it can be verified that [10]

$$\boldsymbol{C}_4 = (m_{4b} - 3\sigma_b^4)\boldsymbol{X}_1 + \sigma_b^4 \boldsymbol{X}_2 + \sigma_b^4 \boldsymbol{I} \tag{22}$$

where

$$\boldsymbol{X}_1 = diag\{\boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1^T, \cdots, \boldsymbol{\alpha}_L \boldsymbol{\alpha}_L^T\}$$

$$\boldsymbol{\alpha}_l^T = [0, \cdots, 0, \underbrace{1}_{l-\text{th element}}, 0, \cdots, 0]_{1 \times L}$$

$$\boldsymbol{X}_2 = [\widetilde{\boldsymbol{X}}_{i,j}]_{L \times L}, \quad \widetilde{\boldsymbol{X}}_{i,j} = \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j^T + \boldsymbol{\alpha}_j \boldsymbol{\alpha}_i^T$$

and L is the length of the input vector $\boldsymbol{b}_n$. Similarly

$$\boldsymbol{C}_5 = \sigma_v^4 \boldsymbol{X}_3 + \sigma_v^4 \boldsymbol{I} \tag{23}$$

where

$$\boldsymbol{X}_3 = [\widetilde{\widetilde{\boldsymbol{X}}}_{i,j}]_{\nu \times \nu}, \;\; \widetilde{\widetilde{\boldsymbol{X}}}_{i,j} = \boldsymbol{\beta}_i \boldsymbol{\beta}_j^T + \boldsymbol{\beta}_j \boldsymbol{\beta}_i^T,$$

$$\boldsymbol{\beta}_l^T = [0, \cdots, 0, \underbrace{1}_{l-\text{th element}}, 0, \cdots, 0]_{1 \times \nu}$$

and ν has been previously defined as the length of the data vector (or the noise vector $\boldsymbol{v}_n$). Substituting (22) in (20) and (23) in (21) respectively, we obtain

$$\boldsymbol{C}_2 = \boldsymbol{C}_6 + \sigma_b^4 \boldsymbol{H}^H \boldsymbol{d}_1 \boldsymbol{d}_2^H \boldsymbol{H}, \tag{24}$$

$$\boldsymbol{C}_3 = unvec\left[\sigma_v^4 \boldsymbol{X}_3 vec(\boldsymbol{d}_1 \boldsymbol{d}_2^H)\right] + \sigma_v^4 \boldsymbol{d}_1 \boldsymbol{d}_2^H \tag{25}$$

where

$$\boldsymbol{C}_6 = unvec\left\{[(m_{4b} - 3\sigma_b^4)\boldsymbol{X}_1 + \sigma_b^4 \boldsymbol{X}_2] vec(\boldsymbol{H}^H \boldsymbol{d}_1 \boldsymbol{d}_2^H \boldsymbol{H})\right\}.$$

Observe that

$$\boldsymbol{R} = \sigma_b^2 \boldsymbol{H}\boldsymbol{H}^H + \sigma_v^2 \boldsymbol{I}, \quad \boldsymbol{\Gamma} = \sigma_b^2 \boldsymbol{I}.$$

Substituting (19), (24), (25) in (18) first, then (18) in (17) we obtain

$$\begin{aligned} \boldsymbol{B}_{1,2} &= \frac{\sigma_b^2 \sigma_v^2}{N}\left(\boldsymbol{H}\boldsymbol{H}^T \boldsymbol{d}_2^* \boldsymbol{d}_1^T + \boldsymbol{d}_2^* \boldsymbol{d}_1^T \boldsymbol{H}^* \boldsymbol{H}^H\right) \\ &+ \frac{\sigma_b^2 \sigma_v^2}{N}\left[(\boldsymbol{d}_2^H \boldsymbol{d}_1)\boldsymbol{H}\boldsymbol{H}^H + (\boldsymbol{d}_2^H \boldsymbol{H}\boldsymbol{H}^H \boldsymbol{d}_1)\boldsymbol{I}\right] \\ &+ \frac{\sigma_v^4}{N}\, unvec\,[\boldsymbol{X}_3\; vec(\boldsymbol{d}_1 \boldsymbol{d}_2^H)] \\ &+ \frac{1}{N}\boldsymbol{H}\boldsymbol{C}_6 \boldsymbol{H}^H \end{aligned} \tag{26}$$

which is our desired result after replacing $\boldsymbol{d}_1$ by $\boldsymbol{d}_{m_1,j_1}$, $\boldsymbol{d}_2$ by $\boldsymbol{d}_{m_2,j_2}$, and $\boldsymbol{B}_{1,2}$ by $\boldsymbol{B}_{m_1,m_2,j_1,j_2}$. $\square$

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