Perturbation Study on MOE-Based Multiuser Detection

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Abstract

The minimum output energy (MOE) receiver has been developed for multiuser detection. Its performance has been shown to be very close to the minimum mean square error (MMSE) receiver at high signal to noise ratio (SNR). However, due to additive noise, the constraint vector required to construct the MOE receiver becomes a biased estimated channel vector. Thus the receiver is also perturbed. In addition to the noise, other factors affect the receiver's performance as well. Effects from such factors as finite number of data samples and channel order overestimation are studied in this paper based on perturbation theory.

1. Introduction

Constrained optimization technique has been successfully applied to multiuser detection in either additive white Gaussian noise (AWGN) channel [2] or multipath fading channel [5], [6], [7]. These minimum output energy (MOE) based approaches minimize the output energy of the receiver subject to certain constraints to guarantee no cancellation of the desired signal. In particular, [7] can efficiently combat multipath channel distortions and provide the optimal constraints. Due to the presence of AWGN, the constraint vector is shown to be a biased estimate for the channel vector and thus some performance loss is induced compared with the minimum mean square error (MMSE) receiver. Detailed study on the noise effect has been performed in [7]. Asymptotic performance of the receiver is derived in a closed form therein in terms of the output signal to interference plus noise ratio (SINR) of the receiver.

Besides the additive noise, there are also other factors which lead to performance degradation. Effects from such factors as finite number of data samples and channel order overestimation are studied in this paper. In practice, only finite number of snapshots (N) are collected for processing. Since the MOE receiver employs the second order statistics of the channel output which are estimated from the received data, the accuracy of correlation estimation depends on the available data record. Therefore due to this error, the optimal constraint vector is perturbed. Finally it will convey an error to the receiver's parameters and result in performance loss. This effect will be investigated analytically in detail although some simulation results are provided in [7]. In the absence of channel order information, it is a common practice to overestimate it. In such a scenario, the associated code filtering matrix in constraint optimization expands its dimension, giving rise to an additional term.

In these cases, perturbation exists in either the data covariance matrix or code filtering matrix. These two quantities are essential for the behavior of the constrained cost function optimized in the MOE method. Once the objective matrix in the cost function gets perturbed, one of its eigenvectors which is the optimal constraint vector is perturbed. Hence the performance of the receiver will degrade. Based on perturbation analysis [1], [4] and the analytical expression for the MOE receiver, the output SINR of the perturbed receiver can be compared with that of the optimal MOE receiver. Closed form expressions will be derived as functions of N and the overestimated channel order respectively. These analytical results will be verified by computer simulations.

2. System Model

Consider a DS-CDMA communication system with J users. User j ($j = 1, \cdots, J$) is assigned spreading codes $c_j(k)$ ($k = 0, \ldots, P-1$) of length P to transmit P chips per information symbol. Let the chip sequence be transmitted through a linear channel with a baseband chip rate discrete-time impulse response $g_j(n)$. Then the received discrete-time signal $y(n)$ at the chip rate receiver is a superposition of the signals from all users plus noise $v(n)$ [7]

$$y(n) = \sum_{j=1}^{J} y_j(n) + v(n),$$

$$y_j(n) = \sum_{l=-\infty}^{\infty} w_j(l) h_j(n - d_j - lP),$$

$$h_j(n) = \sum_{m=-\infty}^{\infty} g_j(m) c_j(n-m) \quad (1)$$

where $w_j(n)$ is zero-mean, i.i.d. information bearing sequence of user j with variance $\sigma_{w_j}^2 = \mathrm{E}\{\|w_j(n)\|^2\}$, $h_j(n)$ is its signature (the convolution of the code with the channel), d_j is the delay of user j in chip periods, and $v(n)$ is assumed to be AWGN with zero-mean and variance $\sigma_v^2 = \mathrm{E}\{\|v(n)\|^2\}$. Assume $0 \leq d_j < P$ and the receiver is synchronized to our desired user - user 1 ($d_1 = 0$). The channel $g_j(n)$ has maximum order $q \ll P$ [7]. After collecting P measurements in a vector, then

$$\boldsymbol{y}(n) = \sum_{j=1}^{J} \boldsymbol{y}_j(n) = \boldsymbol{h}_1 w_1(n) + \boldsymbol{H}_i \boldsymbol{w}_i(n) + \boldsymbol{v}(n) \quad (2)$$

where $\boldsymbol{h}_1 = [h_1(0), \ldots, h_1(P-1)]^T$ is the signature vector of user 1, $\boldsymbol{w}_i(n)$ is the interference vector including intersymbol interference (ISI) and multiuser interference (MUI), $\boldsymbol{H}_i$ is the signature matrix with columns representing signatures of corresponding symbols in $\boldsymbol{w}_i(n)$, $\boldsymbol{v}(n)$ is an AWGN vector. Based on (1), the signature of user 1 can be decomposed as $\boldsymbol{h}_1 = \boldsymbol{C}\boldsymbol{g}_1$ where

$$\boldsymbol{C} = \begin{bmatrix} c_1(0) & & 0 \\ \vdots & \ddots & c_1(0) \\ \vdots & & \vdots \\ c_1(P-1) & \cdots & c_1(P-q-1) \end{bmatrix}, \quad (3)$$

and $\boldsymbol{g}_1$ is the unknown channel vector for user 1

$$\boldsymbol{g}_1^T = [\, g_1(0) \quad \cdots \quad g_1(q) \,]. \quad (4)$$

This structure of the user's signature has been exploited in [7] to design the MOE receiver which will be first briefly reiterated next, in order to analyze the effects from different factors.

3 Constrained Minimum Variance Receiver

A MOE based receiver $\boldsymbol{f}$ is obtained by minimizing the output power subject to parameterized constraints $\boldsymbol{g}$ as [7]

$$\min_{\boldsymbol{f}} \boldsymbol{f}^H \boldsymbol{R} \boldsymbol{f}, \text{ subject to } \boldsymbol{C}^H \boldsymbol{f} = \boldsymbol{g} \quad (5)$$

where $\boldsymbol{R} = \mathrm{E}\{\boldsymbol{y}(n)\boldsymbol{y}^H(n)\}$. For a given $\boldsymbol{g}$, the optimal receiver is $\boldsymbol{f}_o = \boldsymbol{R}^{-1}\boldsymbol{C}(\boldsymbol{C}^H\boldsymbol{R}^{-1}\boldsymbol{C})^{-1}\boldsymbol{g}$, while the minimum output power becomes $\mathcal{J}_{min} = \boldsymbol{g}^H(\boldsymbol{C}^H\boldsymbol{R}^{-1}\boldsymbol{C})^{-1}\boldsymbol{g}$. In [7], it is proposed to optimize the unknown vector $\boldsymbol{g}$ by maximizing $\mathcal{J}_{min}$, that is, by maximizing the energy of the signal component after the interference has been suppressed. It can be transformed into the following problem

$$\boldsymbol{g}_o = \arg\min_{|\boldsymbol{g}|=1} \boldsymbol{g}^H \boldsymbol{A} \boldsymbol{g}, \quad \boldsymbol{A} \triangleq \boldsymbol{C}^H \boldsymbol{R}^{-1} \boldsymbol{C}. \quad (6)$$

Therefore $\boldsymbol{g}_o$ is the eigenvector of $\boldsymbol{A}$ associated with its minimum eigenvalue λ. With $\boldsymbol{g}_o$, the MOE receiver irrespective of the scalar $\frac{1}{\lambda}$ becomes

$$\boldsymbol{f}_o = \boldsymbol{R}^{-1} \boldsymbol{C} \boldsymbol{g}_o \quad (7)$$

This max/min blind approach exhibits near-optimal performance [7] compared with the MMSE receiver.

As observed from (6) and (7), $\boldsymbol{g}_o$ and $\boldsymbol{f}_o$ depend on both $\boldsymbol{R}$ and $\boldsymbol{C}$. If either of them is perturbed, then $\boldsymbol{f}_o$ will be perturbed. We will denote a perturbation by preceding the corresponding quantity by δ, and perturbed quantity with $\tilde{}$

$$\delta \boldsymbol{f} = \tilde{\boldsymbol{f}} - \boldsymbol{f}_o, \quad \delta \boldsymbol{R} = \tilde{\boldsymbol{R}} - \boldsymbol{R}.$$

There exists various perturbation sources. In this paper we only focus on two cases and study the effects of corresponding perturbations: (a) $\boldsymbol{R}$ is estimated from sample average $\tilde{\boldsymbol{R}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{y}(n)\boldsymbol{y}^H(n)$; (b) channel order is overestimated, thus the number of columns in $\boldsymbol{C}$ increases leading to a perturbation in $\boldsymbol{C}$. These perturbations will convey perturbations in $\boldsymbol{A}$, thus incurring performance loss to the MOE receiver.

4 Performance Loss due to Imperfect Estimation

As explained earlier, imperfect estimation of the correlation matrix or channel order results in performance degradation of the receiver due to the inherent nature of the method. We will analyze these losses next.

4.1 Correlation estimation induced performance loss

Perturbation arises in the estimated data correlation matrix when it is estimated from N data vectors [1]. Although $\widehat{\boldsymbol{R}}$ converges to $\boldsymbol{R}$ as $N \to \infty$, a perturbation $\delta\boldsymbol{R}$ due to finite N will cause $\boldsymbol{A}$ perturbed and finally our MOE solution. Here, we are interested in a perturbed eigenvector of $\boldsymbol{A}$ after perturbation in $\boldsymbol{R}$, and investigate how $\delta\boldsymbol{R}$ affects the performance of the receiver.

According to (6), $\boldsymbol{A}$ depends on $\boldsymbol{R}^{-1}$. Under small perturbation assumption (large N) and using Taylor's expansion up to the first order, $(\boldsymbol{R} + \delta\boldsymbol{R})^{-1}$ is approximated by

$$(\boldsymbol{R} + \delta\boldsymbol{R})^{-1} \approx \boldsymbol{R}^{-1} - \boldsymbol{R}^{-1}\delta\boldsymbol{R}\boldsymbol{R}^{-1} \quad (8)$$

Then the first order perturbation of $\boldsymbol{A}$ is

$$\delta \boldsymbol{A} = -\boldsymbol{C}^H \boldsymbol{R}^{-1} \delta \boldsymbol{R} \boldsymbol{R}^{-1} \boldsymbol{C}. \tag{9}$$

Due to $\delta \boldsymbol{A}$, $\boldsymbol{g}_o$ and associated eigenvalue λ are perturbed with perturbations $\delta \boldsymbol{g}_o$ and $\delta\lambda$ respectively. It can be found that [1]

$$\delta \boldsymbol{g}_o \approx -(\boldsymbol{A} - \lambda \boldsymbol{I})^\dagger \delta \boldsymbol{A} \boldsymbol{g}_o. \tag{10}$$

Considering (7) and (8), the perturbation in the MOE receiver has the form

$$\delta \boldsymbol{f} \approx \boldsymbol{R}^{-1} \boldsymbol{C} \delta \boldsymbol{g}_o - \boldsymbol{R}^{-1} \delta \boldsymbol{R} \boldsymbol{R}^{-1} \boldsymbol{C} \boldsymbol{g}_o. \tag{11}$$

Substituting (9) in (10) and then (10) in (11), $\delta \boldsymbol{f}$ is related to a random variable $\delta \boldsymbol{R}$ by

$$\delta \boldsymbol{f} \approx \boldsymbol{A}_1 \delta \boldsymbol{R} \boldsymbol{f}_o \tag{12}$$

where (7) has been used, and

$$\boldsymbol{A}_1 \triangleq [\boldsymbol{R}^{-1} \boldsymbol{C} (\boldsymbol{A} - \lambda \boldsymbol{I})^\dagger \boldsymbol{C}^H - \boldsymbol{I}] \boldsymbol{R}^{-1}$$

The performance of the MOE receiver will be affected by $\delta \boldsymbol{f}$. If we adopt the SINR as a performance measure and define as

$$\widetilde{SINR} \triangleq \sigma_{w_1}^2 \frac{E\{\|\widetilde{\boldsymbol{f}}^H \boldsymbol{h}_1\|^2\}}{E\{\widetilde{\boldsymbol{f}}^H \boldsymbol{R}_i \widetilde{\boldsymbol{f}}\}}, \quad \boldsymbol{R}_i = \boldsymbol{R} - \sigma_{w_1}^2 \boldsymbol{h}_1 \boldsymbol{h}_1^H, \tag{13}$$

then it is perturbed to be

$$\widetilde{SINR} = \sigma_{w_1}^2 \frac{\|\boldsymbol{f}_o^H \boldsymbol{h}_1\|^2 + E\{\delta \boldsymbol{f}^H \boldsymbol{h}_1 \boldsymbol{h}_1^H \delta \boldsymbol{f}\}}{\boldsymbol{f}_o^H \boldsymbol{R}_i \boldsymbol{f}_o + E\{\delta \boldsymbol{f}^H \boldsymbol{R}_i \delta \boldsymbol{f}\}}. \tag{14}$$

Observe that

$$E\{\delta \boldsymbol{f}^H \boldsymbol{h}_1 \boldsymbol{h}_1^H \delta \boldsymbol{f}\} = \boldsymbol{f}_o^H E\{\delta \boldsymbol{R} \boldsymbol{A}_1^H \boldsymbol{h}_1 \boldsymbol{h}_1^H \boldsymbol{A}_1 \delta \boldsymbol{R}\} \boldsymbol{f}_o \tag{15}$$

$$E\{\delta \boldsymbol{f}^H \boldsymbol{R}_i \delta \boldsymbol{f}\} = \boldsymbol{f}_o^H E\{\delta \boldsymbol{R} \boldsymbol{A}_1^H \boldsymbol{R}_i \boldsymbol{A}_1 \delta \boldsymbol{R}\} \boldsymbol{f}_o. \tag{16}$$

To evaluate these two quantities, it suffices to determine

$$\boldsymbol{B} \triangleq E\{\delta \boldsymbol{R} \boldsymbol{D} \delta \boldsymbol{R}\} \tag{17}$$

where $\boldsymbol{D}$ can be replaced by $\boldsymbol{A}_1^H \boldsymbol{h}_1 \boldsymbol{h}_1^H \boldsymbol{A}_1$ or $\boldsymbol{A}_1^H \boldsymbol{R}_i \boldsymbol{A}_1$. Matrix $\boldsymbol{B}$ depends on the second order statistics of $\delta \boldsymbol{R}$. Therefore it will finally rely on up to the fourth order statistics of the received data since $\delta \boldsymbol{R}$ is related to the second order information of the output. It is shown in *Appendix* that for a given data model, statistical properties of the input and additive noise, $\boldsymbol{B}$ can always be evaluated.

According to (37) in *Appendix*, it is clear that $\widetilde{SINR}$ depends on $\frac{1}{N}$. As $N \to \infty$, it converges to the ideal $SINR$. This is not surprising since $\widehat{\boldsymbol{R}}$ will approach $\boldsymbol{R}$. It also depends on the signal power and the noise power.

4.2 Channel order overestimation induced performance loss

When channel order q is overestimated as $\widetilde{q}$, $\boldsymbol{C}$ in (3) is perturbed as $\widetilde{\boldsymbol{C}} = [\boldsymbol{C}, \delta \boldsymbol{C}]$ with dimension $P \times (\widetilde{q} + 1)$. Correspondingly $\boldsymbol{A}$ becomes $\widetilde{\boldsymbol{A}}$

$$\widetilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{A}_2^H \\ \boldsymbol{A}_2 & \boldsymbol{A}_3 \end{bmatrix} \tag{18}$$

$$\boldsymbol{A}_2 \triangleq \delta \boldsymbol{C}^H \boldsymbol{R}^{-1} \boldsymbol{C}, \;\; \boldsymbol{A}_3 \triangleq \delta \boldsymbol{C}^H \boldsymbol{R}^{-1} \delta \boldsymbol{C}.$$

If we introduce a new constraint vector $\widetilde{\boldsymbol{g}} = [\boldsymbol{g}^T \; \delta \boldsymbol{g}^T]^T$ with unit norm constraint on $\boldsymbol{g}$, then from (6) we can solve $\boldsymbol{g}$ and $\delta \boldsymbol{g}$

$$\begin{aligned} (\widehat{\boldsymbol{g}}, \widehat{\delta \boldsymbol{g}}) \;\; &= \;\; \arg \min_{|\boldsymbol{g}|=1} (\boldsymbol{g}^H \boldsymbol{A} \boldsymbol{g} + \delta \boldsymbol{g}^H \boldsymbol{A}_3 \delta \boldsymbol{g} \\ &+ \;\; \delta \boldsymbol{g}^H \boldsymbol{A}_2 \boldsymbol{g} + \boldsymbol{g}^H \boldsymbol{A}_2^H \delta \boldsymbol{g}). \end{aligned} \tag{19}$$

Solving $\delta \boldsymbol{g}$ first,

$$\delta \boldsymbol{g} = -\boldsymbol{A}_3^{-1} \boldsymbol{A}_2 \boldsymbol{g} \tag{20}$$

and substituting it in (19), we obtain

$$\widehat{\boldsymbol{g}} = \arg \min_{|\boldsymbol{g}|=1} \boldsymbol{g}^H (\boldsymbol{A} + \delta \boldsymbol{A}) \boldsymbol{g} \tag{21}$$

where

$$\delta \boldsymbol{A} = -\boldsymbol{A}_2^H \boldsymbol{A}_3^{-1} \boldsymbol{A}_2 \tag{22}$$

According to (10), the perturbation in $\boldsymbol{A}$ causes its eigenvector of interest perturbed

$$\widehat{\boldsymbol{g}} = \boldsymbol{g}_o - (\boldsymbol{A} - \lambda \boldsymbol{I})^\dagger \delta \boldsymbol{A} \boldsymbol{g}_o \tag{23}$$

Then from (20),

$$\widehat{\delta \boldsymbol{g}} = -\boldsymbol{A}_3^{-1} \boldsymbol{A}_2 [\boldsymbol{g}_o - (\boldsymbol{A} - \lambda \boldsymbol{I})^\dagger \delta \boldsymbol{A} \boldsymbol{g}_o] \tag{24}$$

Based on (7), we finally obtain the perturbed receiver vector due to $\delta \boldsymbol{C}$

$$\widetilde{\boldsymbol{f}} = \boldsymbol{R}^{-1} \widetilde{\boldsymbol{C}} \widetilde{\boldsymbol{g}} = \boldsymbol{R}^{-1} (\boldsymbol{C} \widehat{\boldsymbol{g}} + \delta \boldsymbol{C} \widehat{\delta \boldsymbol{g}}) \tag{25}$$

where $\widehat{\boldsymbol{g}}$ and $\widehat{\delta \boldsymbol{g}}$ are given by (23) and (24) respectively. Since in this case $\widetilde{\boldsymbol{f}}$ is deterministic, the SINR has a form (see (13))

$$\widetilde{SINR} = \sigma_{w_1}^2 \frac{\|\widetilde{\boldsymbol{f}}^H \boldsymbol{h}_1\|^2}{\widetilde{\boldsymbol{f}}^H \boldsymbol{R}_i \widetilde{\boldsymbol{f}}}. \tag{26}$$

The effect of channel order overestimation is not so obvious as that of N, since it is embedded in the dimension of $\delta \boldsymbol{C}$ thus $\widehat{\delta \boldsymbol{g}}$. According to (26), the $\widetilde{SINR}$ can be evaluated for different overestimation scenarios.

Effects from both N and channel order overestimation will be tested and compared with our analytical results by simulations.

5 Simulations

We simulate a 5-user CDMA system with equal power and binary inputs. Gold sequence of length 15 is used for each user. Channel coefficients are randomly generated with order 3. The receiver is synchronized to user 1, while other delays are uniformly distributed in $[0, 14]$. Totally 50 independent realizations are performed to obtain the average output SINR. First, the effect of N is tested for a range from 50 to 10^4 and corresponding results are shown in Fig. 1. SNR is set to be $15dB$. Dashed line represents the theoretical value, dashed-dotted line is from our analysis, and solid line is based on simulation. It is observed that when N reaches 10^3, the simulation result converges to the analytical one. This verifies our analysis for large N. These results also converge to their limit as $N \to \infty$. Secondly, when channel order is overestimated in different noisy environments, the SINRs are plotted in Fig. 2. It can be seen that if the estimated order is within certain threshold (e.g., 9 which is about twice of the true order), the method is not so sensitive to order overestimation. But the SINRs may degrade significantly beyond that point.

Appendix: Evaluation of Matrix $\boldsymbol{B}$

For notational convenience, denote $\boldsymbol{y}(n)$ in (2) by $\boldsymbol{y}_n$, and $\boldsymbol{v}(n)$ by $\boldsymbol{v}_n$. We re-write (2) in another form

$$\boldsymbol{y}_n = \boldsymbol{H}\boldsymbol{w}_n + \boldsymbol{v}_n. \tag{27}$$

where

$$\boldsymbol{H} = [\boldsymbol{h}_1, \boldsymbol{H}_i], \;\; \boldsymbol{w}_n = [w_1(n), \boldsymbol{w}_i^T(n)]^T.$$

Substituting $\delta\boldsymbol{R}$ by $\widehat{\boldsymbol{R}} - \boldsymbol{R}$, we obtain

$$\boldsymbol{B} = E\{\widehat{\boldsymbol{R}}\boldsymbol{D}\widehat{\boldsymbol{R}}\} - \boldsymbol{R}\boldsymbol{D}\boldsymbol{R}.$$

Assume N data vectors are mutually independent for our derivation purpose. It can always be made possible by taking only those N vectors which are not contributed by common inputs, although data vectors consecutive in time may be dependent on each other due to channel span. Then

$$E\{\widehat{\boldsymbol{R}}\boldsymbol{D}\widehat{\boldsymbol{R}}\} = \frac{1}{N}\boldsymbol{C}_1 + (1 - \frac{1}{N})\boldsymbol{R}\boldsymbol{D}\boldsymbol{R}$$

where $\boldsymbol{C}_1 \triangleq E\{\boldsymbol{y}_n\boldsymbol{y}_n^H\boldsymbol{D}\boldsymbol{y}_n\boldsymbol{y}_n^H\}$. Therefore

$$\boldsymbol{B} = \frac{1}{N}(\boldsymbol{C}_1 - \boldsymbol{R}\boldsymbol{D}\boldsymbol{R}). \tag{28}$$

According to (27), $\boldsymbol{y}_n\boldsymbol{y}_n^H$ can be expanded to four terms

$$\boldsymbol{y}_n\boldsymbol{y}_n^H = \boldsymbol{H}\boldsymbol{w}_n\boldsymbol{w}_n^H\boldsymbol{H}^H + \boldsymbol{H}\boldsymbol{w}_n\boldsymbol{v}_n^H + \boldsymbol{v}_n\boldsymbol{w}_n^H\boldsymbol{H}^H + \boldsymbol{v}_n\boldsymbol{v}_n^H.$$

Suppose the noise $\boldsymbol{v}_n$ is zero mean and white Gaussian. It is independent of the input sequence which is i.i.d. with zero mean, variance $\boldsymbol{\Gamma}$ and finite fourth-order moment. Since $E\{\boldsymbol{v}_n\boldsymbol{v}_n^H\} = \sigma_v^2\boldsymbol{I}$, $E\{\boldsymbol{w}_n\boldsymbol{w}_n^H\} = \boldsymbol{\Gamma}$. Then only following terms survive in $\boldsymbol{C}_1$

$$\begin{aligned} \boldsymbol{C}_1 &= \boldsymbol{H}\boldsymbol{C}_2\boldsymbol{H}^H + \boldsymbol{C}_3 \\ &+ \boldsymbol{H}E\{\boldsymbol{w}_n\boldsymbol{w}_n^T\}\boldsymbol{H}^T\boldsymbol{D}^T E\{\boldsymbol{v}_n^*\boldsymbol{v}_n^H\} \\ &+ E\{\boldsymbol{v}_n\boldsymbol{v}_n^T\}\boldsymbol{D}^T\boldsymbol{H}^* E\{\boldsymbol{w}_n^*\boldsymbol{w}_n^H\}\boldsymbol{H}^H \\ &+ \sigma_v^2 tr(\boldsymbol{D})\boldsymbol{H}\boldsymbol{\Gamma}\boldsymbol{H}^H + \sigma_v^2 tr(\boldsymbol{D}\boldsymbol{H}\boldsymbol{\Gamma}\boldsymbol{H}^H)\boldsymbol{I} \\ &+ \sigma_v^2\boldsymbol{H}\boldsymbol{\Gamma}\boldsymbol{H}^H\boldsymbol{D} + \sigma_v^2\boldsymbol{D}\boldsymbol{H}\boldsymbol{\Gamma}\boldsymbol{H}^H \end{aligned} \tag{29}$$

where tr denotes the trace of a matrix, superscript $*$ denotes complex conjugate

$$\boldsymbol{C}_2 \triangleq E\{\boldsymbol{w}_n\boldsymbol{w}_n^H\boldsymbol{H}^H\boldsymbol{D}\boldsymbol{H}\boldsymbol{w}_n\boldsymbol{w}_n^H\},$$

$$\boldsymbol{C}_3 \triangleq E\{\boldsymbol{v}_n\boldsymbol{v}_n^H\boldsymbol{D}\boldsymbol{v}_n\boldsymbol{v}_n^H\}.$$

The last four terms in $\boldsymbol{C}_1$ are deterministic quantities. The first four terms are dependent on the statistical properties of the transmitted signal and the noise. Here as an example, we restrict our attention to a typical scenario: the transmitted symbols from different users are taken from real finite alphabets and have equal power, and the noise is also real. For all other cases, one can follow the similar procedures detailed next. However, we will not enumerate them for our concise presentation.

Let us denote the variance of the input by σ_w^2 and its fourth order moment by m_{4w}. Then

$$\boldsymbol{\Gamma} = E\{\boldsymbol{w}_n\boldsymbol{w}_n^T\} = \sigma_w^2\boldsymbol{I}, \quad E\{\boldsymbol{v}_n\boldsymbol{v}_n^T\} = \sigma_v^2\boldsymbol{I}. \tag{30}$$

To easily evaluate $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$, we perform "vec" operations [8] first to combine corresponding terms and then reverse operations "$unvec$" to obtain these matrices

$$\boldsymbol{C}_2 = unvec\,[\boldsymbol{C}_4 vec(\boldsymbol{H}^H\boldsymbol{D}\boldsymbol{H})], \tag{31}$$

$$\boldsymbol{C}_3 = unvec\,[\boldsymbol{C}_5 vec(\boldsymbol{D})] \tag{32}$$

where the property of "vec" has been applied [3], and

$$\boldsymbol{C}_4 \triangleq E\{(\boldsymbol{w}_n\boldsymbol{w}_n^T) \otimes (\boldsymbol{w}_n\boldsymbol{w}_n^T)\},$$

$$\boldsymbol{C}_5 \triangleq E\{(\boldsymbol{v}_n\boldsymbol{v}_n^T) \otimes (\boldsymbol{v}_n\boldsymbol{v}_n^T)\}$$

Then it can be verified that [8]

$$\boldsymbol{C}_4 = (m_{4w} - 3\sigma_w^4)\boldsymbol{X}_1 + \sigma_w^4\boldsymbol{X}_2 + \sigma_w^4\boldsymbol{I} \tag{33}$$

where

$$\boldsymbol{X}_1 = diag\{\boldsymbol{a}_1\boldsymbol{a}_1^T, \cdots, \boldsymbol{a}_L\boldsymbol{a}_L^T\}$$

$$\boldsymbol{a}_l^T = [0, \cdots, 0, \underbrace{1}_{l\text{–th element}}, 0, \cdots, 0]_{1\times L}$$

$$\boldsymbol{X}_2 = [\widetilde{\boldsymbol{X}}_{i,j}]_{L\times L}, \quad \widetilde{\boldsymbol{X}}_{i,j} = \boldsymbol{a}_i\boldsymbol{a}_j^T + \boldsymbol{a}_j\boldsymbol{a}_i^T$$

and L is the length of the input vector $\boldsymbol{w}_n$. Similarly we can evaluate $\boldsymbol{C}_5$. Since the second order and fourth order moments for a white Gaussian process are related, we obtain

$$\boldsymbol{C}_5 = \sigma_v^4 \boldsymbol{X}_3 + \sigma_v^4 \boldsymbol{I} \tag{34}$$

where

$$\boldsymbol{X}_3 = [\widetilde{\widetilde{\boldsymbol{X}}}_{i,j}]_{P\times P}, \quad \widetilde{\widetilde{\boldsymbol{X}}}_{i,j} = \boldsymbol{b}_i \boldsymbol{b}_j^T + \boldsymbol{b}_j \boldsymbol{b}_i^T,$$

$$\boldsymbol{b}_l^T = [0, \cdots, 0, \underbrace{1}_{l-\text{th element}}, 0, \cdots, 0]_{1\times P}$$

and P is the length of the data vector (or the noise vector $\boldsymbol{v}_n$). Substituting (33) in (31) and (34) in (32) respectively, we obtain

$$\boldsymbol{C}_2 = \boldsymbol{C}_6 + \sigma_w^4 \boldsymbol{H}^H \boldsymbol{D} \boldsymbol{H}, \tag{35}$$

$$\boldsymbol{C}_3 = unvec\left[\sigma_v^4 \boldsymbol{X}_3 vec(\boldsymbol{D})\right] + \sigma_v^4 \boldsymbol{D} \tag{36}$$

where

$$\boldsymbol{C}_6 = unvec\left\{[(m_{4w} - 3\sigma_w^4)\boldsymbol{X}_1 + \sigma_w^4 \boldsymbol{X}_2] vec(\boldsymbol{H}^H \boldsymbol{D} \boldsymbol{H})\right\}.$$

Observe that

$$\boldsymbol{R} = \sigma_w^2 \boldsymbol{H}\boldsymbol{H}^H + \sigma_v^2 \boldsymbol{I}.$$

Substituting (30), (35), (36) in (29) first, then (29) in (28) we obtain

$$\begin{aligned}
\boldsymbol{B} &= \frac{\sigma_w^2 \sigma_v^2}{N}\left(\boldsymbol{H}\boldsymbol{H}^T \boldsymbol{D}^T + \boldsymbol{D}^T \boldsymbol{H}^* \boldsymbol{H}^H\right) \\
&+ \frac{\sigma_w^2 \sigma_v^2}{N}\left[tr(\boldsymbol{D})\boldsymbol{H}\boldsymbol{H}^H + tr(\boldsymbol{D}\boldsymbol{H}\boldsymbol{H}^H)\boldsymbol{I}\right] \\
&+ \frac{\sigma_v^4}{N}\, unvec\left[\boldsymbol{X}_3\, vec(\boldsymbol{D})\right] \\
&+ \frac{1}{N}\boldsymbol{H}\boldsymbol{C}_6\boldsymbol{H}^H
\end{aligned} \tag{37}$$

which is our desired result. □

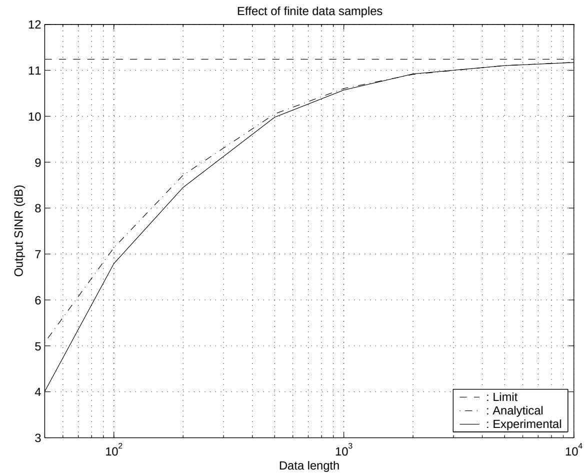

Fig. 1. Data length effect on the output SINR.

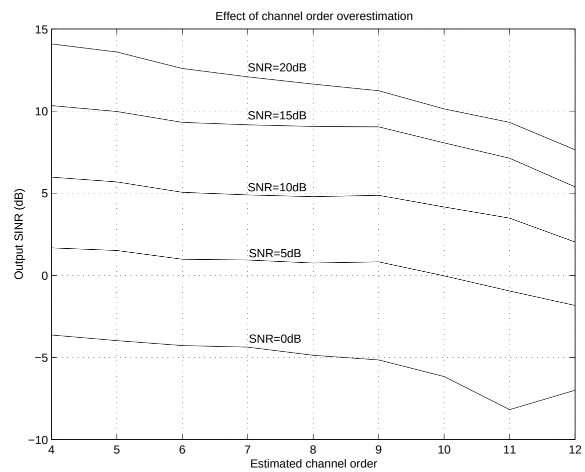

Fig. 2. Effect of channel order overestimation on the output SINR (true order is 3).

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