Clustered Error Correction of Codeword-Stabilized Quantum Codes

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Codeword-stabilized codes are a general class of quantum codes that includes stabilizer codes and many families of nonadditive codes with good parameters. For such a nonadditive code correcting all \(t\)-qubit errors, we propose an algorithm that employs a single measurement to test all errors located on a given set of \(t\) qubits. Compared with exhaustive error screening, this reduces the total number of measurements required for error recovery by a factor of about \(3^t\).

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Quantum computation admits polynomial complexity for many classical algorithms believed to be hard [1,2]. To preserve coherence, quantum computations must be protected by quantum error-correcting codes [3,4]. Stabilizer codes [5] represent an important class of quantum codes that can be encoded and decoded in polynomial time. Recently Refs. [6,7] introduced a larger class of codeword-stabilized (CWS) codes. It includes important code families, such as the stabilizer codes and generally nonadditive union stabilizer (UST) codes [8]. CWS codes have a broader range of code parameters which can be superior to those of any stabilizer code [6–10].

The most important advantage of CWS codes is their close relation with classical codes. In particular, a qubit CWS code \(Q\) can be mapped onto a classical binary code \(C\), with quantum Pauli errors also mapped into some binary error patterns [7]. This way, within the CWS framework, quantum code design can be reduced to classical codes and can employ the wealth of different techniques developed for the latter.

On the other hand, quantum error correction must preserve the original quantum state in all intermediate measurements, and therefore is more restrictive than many classical algorithms. Thus, the design of CWS codes must be complemented by an efficient nondamaging quantum error-correction algorithm. In this Letter, our main goal is to address this important unresolved problem.

We consider a general nonadditive CWS code \((n, K, d)\) of distance \(d\) which encodes \(K\) quantum dimensions into a \(K\)-dimensional subspace of the Hilbert space of \(n\) qubits. This code detects all errors that corrupt up to \((d - 1)\) qubits, and corrects all errors corrupting \(t = [(d - 1)/2]\) or fewer qubits. As a benchmark for our study, we consider generic algorithms that project a corrupted code state into different subspaces. This brute-force technique is similar to the exhaustive error screening in nonlinear classical codes, and requires up to

\[
B(n, t) = \sum_{i=0}^{t} \binom{n}{i} 3^i
\]

measurements to screen all errors of weight \(t\) or less.

To reduce the number of such measurements, we first design an error detection algorithm for USt codes [8]. In the CWS framework, the classical code \(C\) associated with the USt code \(Q = ((n, m2^k, d))\) is decomposed as a group \(C_0\) of \(2^k\) codewords shifted by \(m\) binary “translation” vectors. We prove the following.

**Theorem 1** For a USt code of length \(n\), with a group of size \(2^k\) and dimension \(K = m2^k\), an error-detecting measurement requires no more than \(2m(n - k)(n + 3)\) two-qubit gates.

Then, for a general CWS code \(Q\), we propose an error-correcting method that simultaneously screens all \(4^t\) different errors located on any given subset of \(t\) qubits, by designing an auxiliary USt code which uses binary maps of these errors as generators of the group \(C_0\), and the codewords of the associated classical code \(C\) as translations. This requires only \(\binom{n}{t} - 1\) measurements to screen all groups. Once the corrupted qubits are located, we need up to \(2t\) extra measurements to find the actual error within the group. Overall, this reduces the number \(N(n, t)\) of measurements about \(3^t\) times to

\[
N(n, t) = \binom{n}{t} + 2t - 1.
\]

Our main result is summarized as

**Theorem 2** Consider any \(t\)-error-correcting CWS code of length \(n\) and dimension \(K\). Then this code can correct errors using at most \(N(n, t)\) measurements, each of which requires at most \(2K(n - 1)(n + 3)\) two-qubit gates.

**Definitions.**—Throughout the Letter, we use the Hilbert space \(\mathcal{H}_n\) to represent any \(n\)-qubit state. Also, \(\mathbb{P}_n = \{\pm 1, i\}[I, X, Y, Z]^{\otimes n}\) denotes the Pauli group, where the number of nontrivial terms in the tensor product is the weight of a given \(E \in \mathbb{P}_n\). We say that a space \(\mathcal{P}\) is stabilized by a measurement operator \(M\) with all eigenvalues \(\lambda = \pm 1\) (this includes all Hermitian operators in \(\mathbb{P}_n\)) if \(M|\psi\rangle = |\psi\rangle\) for any state \(|\psi\rangle\) in \(\mathcal{P}\). We will also use the term antisymmetrized if \(M|\psi\rangle = -|\psi\rangle\). A space is stabilized by a set \(\mathcal{M}\) of measurement operators if it is simultaneously stabilized by all operators in \(\mathcal{M}\). A maximal space stabilized by \(\mathcal{M}\) is called the stabilized space \(\mathcal{P}(\mathcal{M})\), and
\(\mathcal{M}\) is called a stabilizer of \(\mathcal{P}(\mathcal{M})\). The corresponding projector is denoted \(\mathcal{P}_M\). The projector \(1 - \mathcal{P}_M\) corresponds to the orthogonal complement \(\mathcal{P}(\mathcal{M})\). For a single measurement operator, \(M = 2\mathcal{P}_M - 1\).

A general quantum code \((n, K, d)\) is a subspace \(Q \subseteq \mathcal{H}_2^n\) of dimension \(K\), such that any detectable error either takes any nonzero state \(|\psi\rangle \in Q\) into a state outside of \(Q\), \(E|\psi\rangle \notin Q\), or acts trivially on \(Q\), \(E|\psi\rangle = C_E|\psi\rangle\) with \(C_E\) independent of \(|\psi\rangle\). A combination \(E_1^e E_2\) of any two errors from a set \(E\) of correctable errors is detectable. The errors are in the same degeneracy class if and only if \(E_1^e E_2\) acts trivially on \(Q\). For a distance-\(d\) code, all Pauli errors of weight up to \((d - 1)\) are detectable, and all Pauli errors of weight up to \(t = \lfloor (d - 1)/2 \rfloor\) are correctable \([2,4]\).

A stabilizer code \([5]\) \((n, k, d)\) is defined as the stabilized space of an Abelian group \(\mathcal{S} = \langle G_1, \ldots, G_{n-k} \rangle\) of size \(2^n\), \(-1 \notin \mathbb{S}\), generated by Hermitian Pauli operators \(G_i\), \(i = 1, \ldots, n-k\). Explicitly, \(Q = \{|\psi\rangle : S|\psi\rangle = |\psi\rangle, \forall S \in \mathbb{S}\}\). The logical operators \(X_i, Z_i, i = 1, \ldots, k\) commute with the code stabilizer \(\mathbb{S}\); they obey the usual Pauli commutation relations. These operators, along with \(G_i \in \mathbb{S}\) and the trivial \(I\), serve as generators of the code normalizer \(N(Q)\), a group of operators \(U \in \mathbb{P}_n\) that preserves the stabilizer \(\mathbb{S}\) under conjugation, \(USU^\dagger = S, U \in N(Q), S \in \mathbb{S}\). Each correctable error \(E \in \mathbb{P}_n\) acting nontrivially on the code anticommutes with at least one generator \(G_i\), and correctable errors in different degeneracy classes anticommute with different subsets of \(\mathbb{S}\). The corrected code \(E(Q) = \{|E|\psi\rangle : |\psi\rangle \in Q\}\) is antistabilized by those generators \(G_i\) that anticommute with \(E\). Thus, a stabilizer code can be corrected by measuring the generators \(G_i\); the corresponding set of eigenvalues \(\lambda_i = \pm 1\) forms the syndrome of the error.

A CWS code \([6,7]\) \((n, K, d)\) is defined in terms of a stabilizer state \(|s\rangle\) (which is an \([n, 0]\) stabilizer code), and a set of \(K\) mutually commuting codeword operators \(\mathcal{W} \equiv \{W_i\}_{i=1}^K \subset \mathbb{P}_n\). Explicitly,

\[
Q = \text{span}\{\{w_i\}_{i=1}, \{w_j\} = W_i|s\rangle\}. \tag{3}
\]

The stabilizer \(\mathcal{S} = \langle S_1, \ldots, S_q \rangle\) of the state \(|s\rangle\) is the maximal Abelian subgroup of the Pauli group such that \(-1 \notin \mathcal{S}\); in the context of CWS codes, it is called word stabilizer \([7]\).

A CWS code is a stabilizer code if and only if the \(K\) word operators \(W_i\) form an (Abelian) group \([7]\). Such a CWS code is called additive; in this case \(K = 2^k\) with integer \(k\).

A USI code \([8]\) can be defined as a CWS code \(Q\) whose word operators contain a group,

\[
\mathcal{W} = \left\{ t_j \prod_{i=1}^k g_i^{\alpha_i} : j = 1, \ldots, m, \alpha_i \in \{0, 1\} \right\}. \tag{4}
\]

Here \(g_i\) are generators of the group \(\mathbb{W} = \langle g_1, \ldots, g_k \rangle\) forming an additive code \(Q_0 = \text{span}\{\mathcal{W}|s\rangle\}_{w \in \mathbb{W}}\) with dimension \(K_0 = 2^k\). The operators \(t_j\) form a set \(\mathcal{S}\) of \(m\) translations for the code \(Q_0\). The translated spaces \(t_j(Q_0)\) are mutually orthogonal, which implies that the dimension of the code \(Q\) is \(K = m2^k\).

The standard form of a CWS code \([6,7]\) is defined in terms of a graph \(G\) with \(n\) vertices and a classical code \(C\) containing \(K\) binary codewords \(c_i\) of length \(n\). The graph adjacency matrix \(R \in \{0, 1\}^{n \times n}\) defines the generators of the stabilizer, \(S_i = X_i^c Z_i^{c_1} Z_i^{c_2} \cdots Z_i^{c_n}\), while the classical codewords define the codeword operators \(W_i = Z_i^{c_1} \cdots Z_i^{c_n}\). Most importantly, the graph relates the error-correction properties \([7]\) of the quantum CWS code \(Q = (G, C)\) and the classical code \(C\). Indeed, the action of a single-qubit error \(X_i\) on the code is equivalent (up to an overall phase) to that of \(X_i S_i = Z_i^{c_1} \cdots Z_i^{c_n}\). Any Pauli operator \(E = Z_i X_i^{c}\) can thus be mapped (up to a phase) to the operator \(Z_i^{c_1} X_i^{c}\) in a standard form \([7]\). For CWS codes in standard form, we will denote the corresponding set of word operators and word stabilizer as \(\mathcal{W}_G\) and \(\mathcal{S}_G\), respectively.

**Exhaustive screening for CWS codes.**—We can detect errors by measuring the operator \(M_G = 2P_Q - I\),

\[
P_Q = \sum_{w \in \mathcal{W}} W|s\rangle\langle s| W^\dagger. \tag{6}
\]

The corresponding ancilla measurement circuit which uses \(2K[n^2 + O(n)]\) two-qubit gates can be constructed as the special case of Eq. (12) below. A different circuit which requires up to \(n^2 + K O(n)\) two-qubit gates is constructed in Ref. \([12]\).

The operators \(E M_G E^\dagger\) stabilize the spaces \(E|Q\rangle\). For a CWS code \(Q\), these spaces are orthogonal for mutually nondegenerate correctable errors \(E\). This implies that an error can be located by measuring such operators for \(E\) from different degeneracy classes. For a \(t\)-error-correcting code we can exhaustively test all correctable errors using up to \(B(n, t)\) measurements [Eq. (1)]. This bound is tight for nondegenerate codes where all linearly independent correctable errors are mutually nondegenerate.

**Measurement algebra.**—To simplify error correction, we will first decompose multiqubit measurements using the algebra of projection operators \([11,13]\). A measurement \(M\) projects a state into the stabilized space \(\mathcal{P}(M)\) or its orthogonal complement. In the following we assume that all measurement operators commute.
In analogy with logical AND, let \( M_1 \land M_0 \) denote the measurement that stabilizes \( \mathcal{P}(M_1) \cap \mathcal{P}(M_0) \). The circuit in Fig. 1 shows an implementation of this combination using logical operations on ancillas. A different circuit which requires only two ancillas is given in Ref. [12].

An operation analogous to logical XOR is defined in terms of the symmetric difference of vector spaces \( A \triangle B = \text{span}(A \setminus B^\perp, B \setminus A^\perp) \). We assume that there exists an orthogonal basis common to all spaces. Then, the symmetric difference \( A \triangle B \triangle C \triangle \ldots \) is spanned by the basis vectors which belong to an odd number of subspaces \( A, B, C, \ldots \). We define the XOR of two commuting measurements, \( M_1 \boxdot M_0 \), as the measurement that stabilizes \( \mathcal{P}(M_1) \triangle \mathcal{P}(M_0) \). The corresponding circuit (Fig. 2) is based on the easy-to-check identity \( M_1 \boxdot M_0 = -M_1M_0 \).

Generally, the equality symbol will denote the equivalence between measurements. If \( M_1M_0 = M_0M_1 \), then

\[
M = M_1 \land M_0 \Leftrightarrow \mathcal{P}(M) = \mathcal{P}(M_1) \cap \mathcal{P}(M_0).
\]

Decomposition of an additive CWS code.—Consider an additive CWS code \( Q_0 \) with the set of word operators \( \mathcal{W}_0 = \{g_1, \ldots, g_k\} \) forming a group. This code is a stabilizer code [7]; it is the common stabilized space of the \( n-k \) generators \( G_i \) of the code stabilizer \( \mathcal{S}_0 \). \( Q_0 = \bigcap_{i=1}^{n-k} \mathcal{P}(G_i) \). According to Eq. (7), we also have

\[
M_0 = M_{Q_0} = \bigwedge_{i=1}^{n-k} G_i,
\]

and can construct the corresponding measurement circuit by analogy with Fig. 1 using associativity. This requires \( 2(n-k) \) controlled-\( n \)-qubit Pauli operators and \( (n-k-1) \) three-qubit Toffoli gates. Adding the corresponding complexities [14], we obtain the overall complexity of up to \( 2(n-k)(n+3) \) two-qubit gates.

This measurement can be done in the basis of the original CWS code. The \( n \) generators \( S_j \in \mathbb{P}_0 \) of the word stabilizer \( \mathcal{S}_0 \) can be chosen [12] to satisfy the orthogonality condition \( S_j^* g_j S_j = (-1)^\delta_{ij} g_j S_j \). Now, the \( k \) logical operators of the code can be chosen as \( \tilde{X}_j = g_j, \tilde{Z}_j = S_j \), and the remaining generators of the orthogonalized word stabilizer can serve as the generators \( G_i = S_{i+k}, i = 1, \ldots, n-k \) of the code stabilizer \( \mathcal{S}_0 \).

Decomposition of a USt code.—Now consider a USt code \( Q \) with the set \( \mathcal{W} \) of word operators in the form (4). Given the generators \( G_i \) of the stabilizer \( \mathcal{S}_0 \) of the additive subcode \( Q_0 \), the generators of the translated code \( t_i Q_0 \) can be written as \( t_i G_i t_i^* \). Then, the corresponding measurement operators [cf. Eq. (9)]

\[
M_j \equiv t_j M_0 t_j^* = \bigwedge_{i=1}^{n-k} t_i G_i t_i^*.
\]

The code \( Q \) is spanned by the orthogonal vector spaces

\[
Q = \mathcal{P}(M_Q) = \text{span}\{\mathcal{P}(M_j)\}_{j=1}^{n-k}, \quad \mathcal{P}(M_i) \perp \mathcal{P}(M_j),
\]

which is equivalent to the symmetric difference \( Q = \mathcal{P}(M_1) \triangle \mathcal{P}(M_2) \triangle \cdots \triangle \mathcal{P}(M_{n-k}) \). According to Eq. (8), this is also equivalent to the decomposition

\[
M_Q = \bigwedge_{j=1}^{n-k} M_j = \bigwedge_{i=1}^{n-k} \left[ \bigwedge_{i=1}^{n-k} t_i G_i t_i^* \right].
\]

Since the XOR (“\( \boxdot \)”) of several measurements is implemented as concatenation [Fig. 2], it requires no overhead; the resulting complexity is then given by Theorem 1.

Clustered measurements for CWS codes.—For a t-error-correcting CWS code \( Q \), consider any subset of correctable errors, \( \mathcal{E}' \subset \mathcal{E} \), and any correctable error \( E \) not degenerate with those in \( \mathcal{E}' \). Then, the space \( \mathcal{E}'(Q) = \text{span}_{E \in \mathcal{E}'} E(Q) \) is orthogonal to \( E(Q) \). Furthermore, errors located on any t qubits (specified by the set of qubit indices \( A = \{i_1, \ldots, i_t\} \) form a group of correctable errors \( \mathcal{E}_A = \langle X_i, Z_i \rangle_{i \in A} \). Thanks to the group property of \( \mathcal{E}_A \), for the set \( \mathcal{E}' = \mathcal{E}_A \), we also have [12] a more restrictive identity \( E(Q) \perp Q_A, \) where \( Q_A = \mathcal{E}_A \mathcal{Q} \). Thus, \( Q_A \) is a quantum code which can detect errors \( E \in \mathcal{E} \) not degenerate with those in \( \mathcal{E}_A \).

Our clustered measurement technique is based on the observation that \( Q_A \) is actually a USt code. Indeed, consider the original CWS code in standard form, \( Q = (G, C) \). The set of operators \( \mathbb{D}_A = \{Z^{C(q)} : E \in \mathcal{E}_A\} \) forms an Abelian group of size \( 2^k = |\mathcal{E}_A| = 2^{2t} \) since the operators \( Z^{C(q)} \) obey the same multiplication table as \( E \in \mathcal{E}_A \) but are not necessarily independent. By construction, different elements of \( \mathbb{D}_A \) are in different error degeneracy classes; therefore, the spaces \( e_i(Q) \) are mutually orthogonal for different \( e_i \in \mathbb{D}_A \). The additional degenerate elements in \( \mathcal{E} \) do not add to the span; therefore, \( Q_A = \mathcal{E}_A Q = \mathcal{D}_A(Q) \). Since \( \mathbb{D}_A \) and \( \mathcal{W}_G \) are combinations of CWS operators only, \( Q_A \) is a USt code in standard form which uses the same stabilizer state \( |s\rangle \) as \( Q \), the Abelian group \( \mathcal{W} = \bigoplus \mathbb{D}_A \), and the codeword operators \( \mathcal{W}_G \) of the code \( Q \) as the translation set \( \mathcal{T} \) [Eq. (4)].
FIG. 2. Measurement for $M_1 \boxplus M_0$. Notations as in Fig. 1. The result $|f\rangle = |1\rangle(Q_1P_0 + P_1Q_0)|\psi\rangle + |0\rangle(P_1P_0 + Q_1Q_0)|\psi\rangle$ is equivalent to $|1\rangle P_{M,G,M_1}|\psi\rangle + |0\rangle(1 - P_{M,G,M_0})|\psi\rangle$.

To form the measurement $M_A \equiv M_{Q_A}$ that stabilizes the USt code $Q_A$, we construct a set of $(n - k)$ orthogonal generators $G_i$ for the additive code $Q_0 \equiv \mathbb{D}_A(\text{span}|s\rangle)$; see Eq. (9). The actual measurement [cf. Eq. (12)],

$$M_A = \boxplus_{W \in \mathcal{W}} \bigwedge_{i=1}^{n-k} (WG_iW^\dagger)$$

satisfies the complexity bound of Theorem 1. The measurement $M_A$ has eigenvalue 1 for all states in $Q_A$, and -1 for all states in $Q_A^\perp$, which corresponds to all correctable errors not degenerate with those in $\mathbb{E}_A$.

To determine the error, we first perform measurements $M_A^t$ for all (but the last one) size-t index sets $A^{(t)}$. After locating the covering set $A$ with Abelian group $\mathbb{D}_A$ of size $|\mathbb{D}_A| = 2^s \leq 2^{2t}$, we can find the error by going over all $s \leq 2t$ subgroups of $\mathbb{E}_A$ with $s - 1$ generators. Each measurement determines whether or not the omitted generator is a part of the error. The error is identified as a product of the generators present in all auxiliary codes that detected no errors. Overall, this requires up to $N(n,t)$ measurements as in Eq. (2). Thus, for any possible CWS code, the former number of $B(n,t)$ measurements [see Eq. (1)] is reduced $B(n,t)/N(n,t)$ times, which exceeds $3^t$ for any $t > 1$.

Some additional acceleration can be gained if the original CWS code is a USt code, with the set of codeword operators (4). In this case, for a given index set $A$, our scheme employs a bigger group $\mathcal{W}$ which includes the generators of both $\mathbb{D}_A$ and the original group $\mathbb{W}$, and a smaller translation set $\mathcal{T}$ of size $m < K$. The complexity of a single measurement would then be reduced to $2mn^2$, compared to $2Kn^2$ in Theorem 2. Screening of $N(n,t)$ or fewer qubit clusters will locate the error.

Note also that in the special case of stabilizer codes, our error-grouping technique is equivalent to the syndrome-based recovery [12]. Indeed, for a stabilizer code $Q = \{[n,k,d]\}$, the degeneracy classes form an Abelian group $\mathbb{E} = \{e_1, \ldots, e_{n-k}\}$ whose $2^{n-k}$ elements are enumerated by different syndromes [15]. To locate the error, we can go over all $(n - k)$ USt codes $\mathbb{E}_a(Q)$ generated by the subgroups of $\mathbb{E}$ with one generator, $e_a$, missing. Then, the code $\mathbb{E}_a(Q)$ is a stabilizer code that has to correct only one nontrivial error, $E_a = \{e_a\}$. The corresponding stabilizer $\mathbb{S}_a$ has only one generator. Thus, error can be located by independent measurements of $n - k$ Pauli operators, as we do to measure the syndrome.

In conclusion, we constructed an accelerated clustered quantum error-correction algorithm for a nonadditive CWS code which uses a set of auxiliary USt codes associated with groups of correctable errors on size-t clusters. For a generic nonadditive code, this reduces the number of error-correcting measurements approximately $3^t$ times, compared to exhaustive screening of all correctable errors of weight $t$ and smaller.

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[15] This group is the quotient group of the Abelian version of $P_a$ which ignores the phases, by the Abelian version of the code normalizer $\mathbb{N}(Q)$.