Blind Deconvolution

• Consider the convolution

\[ u(n) = \sum_{k=-\infty}^{\infty} h_k x(n-k) \]

where both \( h_k \) and \( x(n) \) are unknown.

• How can we find \( x(n) \) from \( u(n) \)?

Bussgang Algorithm

• Assumptions:

\[
E[x(n)] = 0 \\
E[x(n)x(k)] = \delta(n-k) \\
x(n) \text{ is real}
\]

There is \( w_k \) such that

\[
\sum_{k=-\infty}^{\infty} w_k h_{n-k} = \delta(n)
\]

• Apply linear equalizer:

\[
y(n) = \sum_{i=-L}^{L} \hat{w}_i(n) u(n-i) = x(n) + v(n)
\]

where

\[
v(n) = \sum_{i=-\infty}^{\infty} \left[ \hat{w}_i(n) - w_i \right] u(n-i)
\]
• We can rewrite

\[ v(n) = \sum_{i=-\infty}^{\infty} [\hat{w}_i(n) - w_i] u(n - i) \]

\[ = \sum_{i=-\infty}^{\infty} [\hat{w}_i(n) - w_i] \sum_{k=-\infty}^{\infty} h_k x(n - i - k) \]

\[ = \sum_{l=-\infty}^{\infty} x(l) \Delta(n - l) \]

where

\[ \Delta(n) = \sum_{k=-\infty}^{\infty} h_k [\hat{w}_{n-k}(n) - w_{n-k}] \]

• It can be shown that

\[ E[v(n)] = 0 \]

\[ E[v(n)v(n-j)] = \sum_{l=-\infty}^{\infty} \Delta(n - l) \Delta(n - l - j) \]

\[ = \begin{cases} \sigma^2(n) \approx \sum_{l=-\infty}^{\infty} \Delta^2(n - l), & j = 0 \\ 0, & j \neq 0 \end{cases} \]

\[ E[x(n)v(n-j)] = \Delta(-j) << \sigma^2(n) \]

• Furthermore, we can assume that \( v(n) \) is Gaussian.
• We now compute the MMSE estimate of \( x(n) \) given \( y(n) \), i.e.,
\[
\hat{x}(n) = E(x(n)|y(n))
\]
\[
= \int_{-\infty}^{\infty} x f_{x(n)|y(n)}(x|y(n)) dx
\]
\[
= \frac{\int_{-\infty}^{\infty} x f_{x(n)|y(n)}(y(n)|x) f_{x(n)}(x) dx}{f_{y(n)}(y(n))}
\]
which is in general a nonlinear function, i.e.,
\[
\hat{x}(n) = g(y(n))
\]

• Can you work out the density functions required to compute \( \hat{x}(n) = g(y(n)) \)?

• Once \( \hat{x}(n) \) is computed at each \( n \), we can use the LMS algorithm to update the weight:
\[
\hat{w}_i(n+1) = \hat{w}_i(n) + \mu u(n-i) e(n), \quad i = 0, \pm 1, \pm 2, \ldots
\]
with
\[
e(n) = \hat{x}(n) - y(n)
\]

• This is the so-called Bussgang algorithm. But its convergence property is uncertain. Why?
• When convergent, we know

\[ E(u(n - i)e(n)) = 0 \]

That is,

\[ E(u(n - i)y(n)) = E(u(n - i)g(y(n))) \]

\[ E\left(y(n) \sum_{i=-L}^{L} \hat{w}_{i-k}(n)u(n - i)\right) = E\left(g(y(n)) \sum_{i=-L}^{L} \hat{w}_{i-k}(n)u(n - i)\right) \]

But, for large \( L \),

\[ y(n - k) = \sum_{i=-L}^{L} \hat{w}_{i}(n)u(n - k - i) \]

\[ = \sum_{i=-L+k}^{L+k} \hat{w}_{i-k}(n)u(n - i) \]

\[ \approx \sum_{i=-L}^{L} \hat{w}_{i-k}(n)u(n - i) \]

Then we have

\[ E(y(n)y(n - k)) \approx E(g(y(n))y(n - k)) \]

Bussgang equation
• Complex Bussgang Algorithm:

If $x(n)$ is complex, then …

• Decision-Directed Algorithm:

$$\hat{x}(n) = g(y(n)) = \text{decision}$$

Example:

$$\hat{x}(n) = \begin{cases} 
1 & y(n) > 0 \\
-1 & y(n) < 0 
\end{cases}$$

• Sato Algorithm:

$$\hat{x}(n) = \gamma \text{sgn}[y(n)]$$

where

$$\gamma = \frac{E\left(x^2(n)\right)}{E[|x(n)|]}$$

• Godard Algorithm:

Cost function: $J(n) = E\left(\left(|y(n)|^p - R_p\right)^2\right)$

Where $R_p = \frac{E(|x(n)|^{2p})}{E(|x(n)|^p)}$ obtained from

$$\frac{\partial}{\partial w} J(n) \bigg|_{|y(n)|=|x(n)|} = 0$$
Learning rule:

$$w(n + 1) = w(n) + \mu u(n)e^*(n)$$

where

$$e(n) = y(n)|y(n)|^{p-2}\left(R_p - |y(n)|^p\right)$$

If $p = 2$, it is the standard constant modulus algorithm (CMA).
Channel Estimation Using Polyspectra

• Recall the input-output relationship in time-domain:

\[ u(n) = \sum_{k=-\infty}^{\infty} h_k x(n - k) \]

• Input-output relationship in power spectrum:

\[ S_u(f) = |H(f)|^2 S_x(f) \]

where

\[ H(f) = \alpha \prod_{i} \left( 1 - a_i z^{-1} \right) \prod_{j} \left( 1 - b_j z^{-1} \right) \bigg|_{z = \exp(j2\pi f)} \]

• Note: Power spectrum is blind to the phase of \( H(f) \).

• Higher-order statistics:

\[ M(\lambda_1, \lambda_2, \ldots, \lambda_k) = E \left( \exp \left( \sum_{i=1}^{k} \lambda_i u(n + \tau_i) \right) \right) \]

Moment-generating function

\[ K(\lambda_1, \lambda_2, \ldots, \lambda_k) = \ln E \left( \exp \left( \sum_{i=1}^{k} \lambda_i u(n + \tau_i) \right) \right) \]

Cumulant-generating function
\[ m(\tau_1, \tau_2, \ldots, \tau_k) = \frac{\partial^k}{\partial \lambda_1 \partial \lambda_2 \ldots \partial \lambda_k} M(\lambda_1, \lambda_2, \ldots, \lambda_k) \bigg|_{\lambda_1=\lambda_2=\ldots=0} \]

\[ = E(u(n + \tau_1)u(n + \tau_2)\cdots u(n + \tau_k)) \]

\text{kth-order moment}

\[ c(\tau_1, \tau_2, \ldots, \tau_k) = \frac{\partial^k}{\partial \lambda_1 \partial \lambda_2 \ldots \partial \lambda_k} K(\lambda_1, \lambda_2, \ldots, \lambda_k) \bigg|_{\lambda_1=\lambda_2=\ldots=0} \]

\text{kth-order cumulant}

- If \( u(n) \) is stationary,

\[ m(\tau_1, \tau_2, \ldots, \tau_k) = m(\tau_1 + \tau, \tau_2 + \tau, \ldots, \tau_k + \tau) \]

\[ c(\tau_1, \tau_2, \ldots, \tau_k) = c(\tau_1 + \tau, \tau_2 + \tau, \ldots, \tau_k + \tau) \]

- We will write

\[ m_k(\tau_1, \tau_2, \ldots, \tau_{k-1}) \equiv m(\tau, \tau + \tau_1, \ldots, \tau + \tau_{k-1}) \]

\[ c_k(\tau_1, \tau_2, \ldots, \tau_{k-1}) \equiv c(\tau, \tau + \tau_1, \ldots, \tau + \tau_{k-1}) \]

- It can be shown that if \( u(n) \) has zero mean,

\[ c_2(\tau) = m_2(\tau) \]

\[ c_3(\tau_1, \tau_2) = m_3(\tau_1, \tau_2) \]

\[ c_4(\tau_1, \tau_2, \tau_3) = m_4(\tau_1, \tau_2, \tau_3) - m_2(\tau_1)m_2(\tau_3 - \tau_2) \]

\[ - m_2(\tau_2)m_2(\tau_1 - \tau_3) - m_2(\tau_3)m_2(\tau_1 - \tau_2) \]
• If $u(n)$ is zero-mean Gaussian,
  \[ c_{u,k}(\tau_1, \tau_2, \ldots, \tau_{k-1}) = 0 \text{ for } k > 2. \]

• If the input is i.i.d.,
  \[ c_{u,k}(\tau_1, \tau_2, \ldots, \tau_{k-1}) = \gamma_k \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} \ldots h_{i+\tau_{k-1}} \]
  \[ \gamma_k \triangleq c_{x,k}(\tau_1, \tau_2, \ldots, \tau_{k-1}) \big|_{\tau_1 = \tau_2 = \ldots = \tau_{k-1} = 0} \]

• If $x(n)$ has a symmetric distribution,
  \[ \gamma_3 = 0 \text{ (skewness)} \]
  \[ \gamma_4 > 0 \text{ (kurtosis) only if } x(n) \text{ is nonGaussian} \]

• Polyspectra:
  \[ C_2(\omega) = F(c_2(\omega)) \text{ power spectrum} \]
  \[ C_3(\omega_1, \omega_2) = F(c_3(\tau_1, \tau_2)) \text{ bispectrum} \]
  \[ C_4(\omega_1, \omega_2, \omega_3) = F(c_4(\tau_1, \tau_2, \tau_3)) \text{ trispectrum} \]
  \[ k_4(\tau_1, \tau_2, \tau_3) = F^{-1}(\ln C_4(\omega_1, \omega_2, \omega_3)) \text{ tricepsrtum} \]

• Given the previous model, we have
  \[ C_{u,4}(\omega_1, \omega_2, \omega_3) = \gamma_{x,4} H(e^{j\omega_1}) H(e^{j\omega_2}) H(e^{j\omega_3}) H(e^{-j(\omega_1+\omega_2+\omega_3)}) \]