Ideal All-Pass Systems with Linear Phase

- The frequency response:
  \[ H(e^{j\omega}) = e^{-j\omega\alpha}, \text{ for } |\omega| < \pi \]
  where \( \alpha \) may or may not be an integer

- The impulse response:
  \[ h(n) = \frac{\sin(\pi(n - \alpha))}{\pi(n - \alpha)} \]
  \[ h(n) = \delta(n - \alpha) \text{ if } \alpha \text{ is an integer} \]

- If \( x(n) = x_c(nT) \) and the Nyquist sampling rate is used, then
  \[ y(n) = h(n) * x(n) = x_c(nT - \alpha T) \]

Linear Systems with Generalized Linear Phase

- Constant group delay:
  \[ H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta} \]
  where \( A(e^{j\omega}) \) is real valued, and
  the group delay is \( \alpha \)
• If \( x(n) = x_c(nT) \), \( X_c(j\Omega) \neq 0 \) only for

\[ |\Omega| < \Omega_c < \frac{\pi}{T}, \quad A(e^{j\omega}) = A_c \text{ for } |\omega| < T\Omega_c, \text{ then} \]

\[ y(n) = x(n) * h(n) = A_c x_c(nT - \alpha T) e^{j\beta} \]

**FIR Linear Phase Systems**

• Type I:

\[ h(n) = h(M - n) \text{ for } 0 \leq n \leq M \]

\[ M \text{ is even} \]

*What is the frequency response?*

• Type II:

\[ h(n) = h(M - n) \text{ for } 0 \leq n \leq M \]

\[ M \text{ is odd} \]

*What is the frequency response?*

Type III:

\[ h(n) = -h(M - n) \text{ for } 0 \leq n \leq M \]

\[ M \text{ is even} \]

*What is the frequency response?*
Type IV:

\[ h(n) = -h(M - n) \quad \text{for} \quad 0 \leq n \leq M \]

\( M \) is odd

*What is the frequency response?*

**Zeros of FIR Linear-Phase Systems**

- If \( h(n) \) is real valued, then
  \[
  H(z) = \sum_{n} h(n)z^{-n} = H^*(z^*)
  \]
  so, the complex zeros are in *conjugate pairs*

- If \( h(n) \) is symmetric or antisymmetric, then
  \[
  H(z) = \sum_{n=0}^{M} h(n)z^{-n} = \pm z^{-M} H(z^{-1})
  \]
  so, all zeros (except \( \pm 1 \)) are in *reciprocal pairs*

- If \( h(n) \) is real and symmetric/antisymmetric, then
  all complex zeros are in *conjugate reciprocal quadruplets*
• At $z = -1$,

\[ H(-1) = (-1)^{-M} H(-1) \text{ for symmetric } h(n) \]
\[ H(-1) = (-1)^{-M+1} H(-1) \text{ for anti-symmetric } h(n) \]

• Hence, $z = -1$ must be a zero if

$h(n)$ is symmetric and $M$ is odd, or
$h(n)$ is anti-symmetric and $M$ is even.

• At $z = 1$,

\[ H(1) = H(1) \text{ for symmetric } h(n) \]
\[ H(1) = -H(1) \text{ for anti-symmetric } h(n) \]

• Hence, $z = 1$ must be a zero if

$h(n)$ is anti-symmetric.

Factorization of FIR Linear-Phase Systems

• For any FIR linear-phase system,

\[ H(z) = H_{\min}(z)H_{uc}(z)H_{\max}(z) \]

*minimum phase factor vs maximum phase factor*