The Z-Transform

• (Double-sided) Z-transform:

\[ X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \]

More general than Fourier transform

More convenient in some analysis

• Region of convergence (ROC):

\[ ROC = \{z : X(z) < \infty\} \]

• Example - right-sided exponential sequence:

\[ x(n) = a^n u(n) \]

\[ X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} \]

ROC: \(|z| > |a|\)

• Example – left-sided exponential sequence:

\[ x(n) = -a^n u(-n-1) \]

\[ X(z) = -\sum_{n=-\infty}^{-1} (az^{-1})^n = ... = \frac{1}{1 - az^{-1}} \]

ROC: \(|z| < |a|\)
• Example – \textit{double-sided exponential sequence}:

\[
x(n) = a^n u(n) + b^n u(n) - c^n u(-n-1) - d^n u(-n-1)
\]

\[
X(z) = \cdots = \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} + \frac{1}{1-cz^{-1}} + \frac{1}{1-dz^{-1}}
\]

\text{ROC: max}(|a|, |b|) < |z| < \min(|c|, |d|)

• Example – \textit{finite-length sequence}:

\[
x(n) = \begin{cases} 
a^n & 0 \leq n \leq N-1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \frac{1 - (az^{-1})^N}{1 - (az^{-1})}
\]

\[
= \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}
\]

\text{ROC: } |z| > 0

\text{zeros: } z_k = ae^{j2\pi k / N}, \ k = 1, 2, \ldots, N

\text{pole: } z = 0 \text{ of order } N-1
Properties of ROC of the Z-Transform

- ROC is a ring or disk centered at the origin
- Fourier transform converges absolutely iff ROC includes the unit circle
- ROC contains no poles but is bounded by poles
- If the sequence is finite in length, ROC is the entire $z$-plane except possibly $z = 0$ and $z = \infty$.
- If the sequence is right-sided, ROC is an outer disk
- If the sequence is left-sided, ROC is an inner disk
- If the sequence is double-sided, ROC is a ring
- ROC is a connected region.
The Inverse Z-Transform

- Formal approach:

\[ x(n) = \frac{1}{j2\pi} \oint_{C \in \text{ROC}} X(z)z^{n-1}dz \]

contour integral

- Inverse Fourier transform: If ROC contains the unit circle, then

\[ x(n) = \frac{1}{j2\pi} \oint_{|z|=1} X(z)z^{n-1}dz \]

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} X(e^{j\omega})e^{j\omega n} d\omega \]

- Inspection method (table lookup):

\[ x(n) \xrightarrow{Z} X(z) \text{ with a ROC} \]

see Table 3.1
• Partial fraction expansion: Given the rational function

\[ X(z) = \sum_{k=0}^{M} b_k z^{-k} \]
\[ X(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \]

we can write

\[ X(z) = b_0 \prod_{k=1}^{M} (1 - c_k z^{-1}) \]
\[ X(z) = a_0 \prod_{k=1}^{N} (1 - d_k z^{-1}) \]

\[ = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \]

The first term exists if \( M \geq N \), and can be obtained via *long division*;

The second term holds if all poles are *simple* (of order one), and the coefficients follow from:

\[ A_k = \left\{ (1 - d_k z^{-1})X(k) \right\}_{z=d_k} \]
Example:

\[ X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \]

\[ = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \]

\[ = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}} \]

If ROC is \(|z| > 1\), then

\[ x(n) = 2\delta(n) - 9\left(\frac{1}{2}\right)^n u(n) + 8u(n) \]

If ROC is \(\frac{1}{2} < |z| < 1\), then

\[ x(n) = 2\delta(n) - 9\left(\frac{1}{2}\right)^n u(n) - 8u(-n - 1) \]

If ROC is \(|z| < \frac{1}{2}\), then

\[ x(n) = 2\delta(n) + 9\left(\frac{1}{2}\right)^n u(-n - 1) - 8u(-n - 1) \]
• Power series expansion:

\[ X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \]

\[ = \ldots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + \ldots \]

• Example of long division:

\[ X(z) = \frac{1}{1 - az^{-1}} = \ldots \]

*decreasing power of z*

*for causal sequence*

\[ X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{-a + z} = \ldots \]

*increasing power of z*

*for anticausal sequence*
Properties of the Z-Transform

• Linearity

• Time shifting: \( x(n - n_0) \xrightarrow{Z} z^{-n_0} X(z) \)

• Multiplication by exponential:
  \[ z^n_0 x(n) \xrightarrow{Z} X(z / z_0) \]
  ROC: \(|z_0|R_x\)

• Differentiation of Z-transform:
  \[ nx(n) \xrightarrow{Z} -z \frac{dX(z)}{dz} \]
  ROC: unchanged

• Conjugation of a complex sequence:
  \[ x^*(n) \xrightarrow{Z} X^*(z^*) \]
  ROC: unchanged

• Time reversal
  \[ x^*(-n) \xrightarrow{Z} X^*(1 / z^*) \]
  ROC: \(1 / R_x\)

• Convolution
  \[ x(n) * y(n) \xrightarrow{Z} X(z)Y(z) \]
  ROC: \(R_x \cap R_y\)

• Initial-value theorem: if the sequence is causal,
  \[ x(0) = \lim_{z \to \infty} X(z) \]