Properties of LTI Systems

- Commutative:
  \[ x(n) * h(n) = h(n) * x(n) \]

- Linearity:
  \[ x(n) *[ h_1(n) + h_2(n) ] = x(n) * h_1(n) + x(n) * h_2(n) \]

- Cascade connection:
  \[ y(n) = [x(n) * h_1(n)] * h_2(n) = x(n) *[ h_1(n) * h_2(n) ] \]

  *Can you sketch the diagram?*

- Parallel connection:
  \[ y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)] = x(n) *[ h_1(n) + h_2(n) ] \]

  *Can you sketch the diagram?*

- Stable LTI system:
  \[ \sum_{k=-\infty}^{\infty} |h(k)| < \infty \]

  *sufficient and necessary*
• Causal LTI system:

\[ h(n) = 0 \quad \text{for } n < 0 \]

\textit{sufficient and necessary}

**Linear Constant-Coefficient Difference Equation**

• The general form:

\[
\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)
\]

• Recursive computation:

\[
y(n) = -\frac{1}{a_0} \sum_{k=1}^{N} a_k y(n-k) + \frac{1}{a_0} \sum_{m=0}^{M} b_m x(n-m)
\]

• Accumulator:

\[
y(n) = \sum_{k=-\infty}^{n} x(k)
\]

or equivalently,

\[
y(n) - y(n-1) = x(n)
\]

\textit{Can you sketch the diagram?}
• Moving average (MA):

\[ y(n) = \sum_{m=0}^{M} b_m x(n-m) \]

where \( b_m \)'s are the weights

• Auto regression (AR):

\[ \sum_{k=0}^{N} a_k y(n-k) = x(n) \]

• Homogeneous equation:

\[ \sum_{k=0}^{N} a_k y(n-k) = 0 \]

where the input is zero!

The output may not be zero as

\[ y(n) = \sum_{m=1}^{N} A_m z_m^n \]

if \( \sum_{k=1}^{N} a_k z_m^{-k} = 0 \)

The output is zero if

\[ y(-1) = y(-2) = \ldots = y(-N) = 0 \]
• The ARMA system:

\[ \sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m) \]

is linear, time-invariant, and causal if

\[ y(-1) = y(-2) = ... = y(-N) = 0 \]

zero initial conditions

• With the zero initial conditions, the ARMA system can be uniquely represented by its impulse response \( h(n), 0 < n < \infty \).
Frequency-Domain Representation of LTI Systems

• Eigenfunction:
  If $x(n) = \exp(j\omega n)$, $-\infty < n < \infty$, then,

  $$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

  $$= \sum_{k=-\infty}^{\infty} h(k)e^{j\omega n - j\omega k}$$

  $$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

  $$= e^{j\omega n} H(e^{j\omega})$$

  Hence, $\exp(j\omega n)$ is called the eigenfunction, and $H(j\omega)$ is called the frequency response.

• Generally, we can write

  $$x(n) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\omega_k n}$$

  Then, by linearity,

  $$y(n) = \sum_{k=-\infty}^{\infty} \alpha_k H(e^{j\omega_k})e^{j\omega_k n}$$
• Properties of frequency response:
  \[ H(j \omega) \] is complex
  \[ H(j \omega) \] is periodic

• Ideal delay:
  \[ y(n) = x(n - n_0) \]
  \[ h(n) = \delta(n - n_0) \]
  \[ H(j \omega) = e^{-j \omega n_0} \]

• Ideal frequency-selective filters:
  \[ |H(j \omega)| = \begin{cases} 1 & \omega \in \Omega_0 \\ 0 & \text{otherwise} \end{cases} \]