Analysis of Positioning Uncertainty in Simultaneous Localization and Mapping (SLAM)

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Abstract

This Technical Report studies the time evolution of the covariance of the position estimates in single-robot Simultaneous Localization And Mapping (SLAM). A closed-form expression is derived, that establishes a functional relation between the noise parameters of the robot's proprioceptive and exteroceptive sensors, the number of features being mapped, and the attainable accuracy of SLAM. Furthermore, it is demonstrated how prior information about the spatial density of landmarks can be utilized in order to compute a tight upper bound on the expected covariance of the positioning errors. The derived closed-form expressions enable the prediction of SLAM positioning performance, without resorting to extensive simulations, and thus offer an analytical tool for determining the sensor characteristics required to achieve a desired degree of accuracy.

1 Motion in 1-D

We first examine the case of a robot moving in a one-dimensional environment, observing the positions of N landmarks that exist in the same one-dimensional world. This is a simplified case that will allow us to develop intuition for the more complicated (non-linear) case of motion in two dimensions.

1.1 Continuous time Riccati Equation

In order to describe the uncertainty of the position estimates for the robot and the landmarks during SLAM, we formulate the problem as the estimation of the state of a linear dynamic system, and use the Riccati differential equation (Eq. (100)) to describe the covariance of the estimates.

In Appendix E the general formula for the state propagation, measurement, and Riccati equations are given. For a system comprising of a robot and N landmarks in 1-D, the $(N+1) \times 1$ state vector is $x = [x_R \;\; x_L]^T$, where x_R is the position of the robot, and x_L is the $N \times 1$ vector containing the positions of the landmarks. The landmarks are assumed stationary, and the robot uses its velocity measurements to propagate its state estimates. Thus the state propagation equation is

$$\dot{x}(t) = v(t) + Gw(t)$$

where $v(t) = [v_R(t) \;\; \mathbf{0}_{1\times N}]^T$ is the input vector[1] $w(t)$ is the noise in the measurement of the velocity of the robot, assumed to be white zero-mean Gaussian, with constant variance q, and $G = [1 \;\; \mathbf{0}_{1\times N}]^T$. The state transition matrix is equal to $F(t) = \mathbf{0}_{(N+1)\times(N+1)}$.

[1]Throughout this Technical Report, $\mathbf{0}_{m\times n}$ denotes the $m \times n$ matrix of zeros, $\mathbf{1}_{m\times n}$ denotes the $m \times n$ matrix of ones, and $I_{n\times n}$ denotes the $n \times n$ identity matrix.

The robot is also equipped with an exteroceptive sensor enabling it to measure the relative positions of the landmarks at each time instant. The relative position measurement associated with landmark i is given by the equation

$$
\begin{aligned}
z_i(t) &= x_{L_i}(t) - x_R(t) \\
&= \begin{bmatrix} -1 & 0 & \dots & \underbrace{1}_{i+1 \text{ position}} & \dots & 0 \end{bmatrix} x(t) \\
&= H_i x(t)
\end{aligned}
$$

and thus the measurement matrix for the system is found by stacking the rows H_i, $i = 1 \dots N$:

$$H = \begin{bmatrix} -\mathbf{1}_{N\times 1} & I_{N\times N} \end{bmatrix} \tag{1}$$

The measurement model for the exteroceptive measurements is:

$$z(t) = Hx(t) + n(t) \tag{2}$$

where $n(t)$ is the measurement noise, assumed white zero-mean Gaussian, with covariance matrix R.

The Riccati differential equation that describes the time evolution of the covariance of the position estimates of the robot and landmarks is (cf. Appendix E):

$$
\begin{aligned}
\dot{P}(t) &= GqG^T - P(t)H^T R^{-1} H P(t) \\
&= qQ_n - P(t)H^T R^{-1} H P(t)
\end{aligned} \tag{3}
$$

where $P(t)$ denotes the covariance matrix, and

$$Q_n = \begin{bmatrix} 1 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times N} \end{bmatrix} \tag{4}$$

For the solution of this matrix differential equation the standard methodology involving the decomposition of $P(t)$ into two matrices, and forming the Hamiltonian matrix is employed [1]. The solution is described in what follows.

In order to facilitate the derivations, we first define as P_n the *normalized covariance*

$$P_n(t) = \frac{1}{q} P(t) \Rightarrow P(t) = qP_n(t) \tag{5}$$

Substitution in Eq. (3) yields

$$
\begin{aligned}
q\dot{P}_n(t) &= qQ_n - qP_n(t)H^T R^{-1} H q P_n(t) \Rightarrow \\
\dot{P}_n(t) &= Q_n - P_n(t)(qH^T R^{-1} H) P_n(t)
\end{aligned}
$$

The notation is simplified by introducing the matrix $C = qH^T R^{-1} H$, yielding the following Riccati differential equation:

$$\dot{P}_n(t) = Q_n - P_n(t) C P_n(t) \tag{6}$$

The initial value of this differential equation is denoted as

$$P_n(0) = \begin{bmatrix} P_{rr} & P_{rL} \\ P_{Lr} & P_{LL} \end{bmatrix} \tag{7}$$

which implies that the initial value of the covariance matrix for 1D SLAM is equal to

$$P(0) = qP_n(0) = q \begin{bmatrix} P_{rr} & P_{rL} \\ P_{Lr} & P_{LL} \end{bmatrix} \tag{8}$$

This is the most general form possible, since no assumption on $P(0)$ is imposed.

The structure of matrix C will be useful in the following. It is easy to see that C is a $(N+1) \times (N+1)$ matrix, given by

$$C = H^T R^{-1} H = \begin{bmatrix} \rho^2 & -\mathbf{r}^T \\ -\mathbf{r} & R_n^{-1} \end{bmatrix} \tag{9}$$

where

$$\rho^2 = \mathbf{1}_{1\times N} R_n^{-1} \mathbf{1}_{N\times 1},$$

$$R_n = \frac{1}{q} R$$

and

$$\mathbf{r} = R_n^{-1} \mathbf{1}_{N\times 1}$$

At this point we note that the measurement matrix H can be viewed as the incidence matrix of a directed graph with $N+1$ vertices, in which N vertices (corresponding to the landmarks) are connected with exactly one edge to a vertex corresponding to the robot. Since this is a connected graph, the rank of its incidence matrix is N [2]. Moreover, R is a covariance matrix, and therefore it is of full rank. Consequently, the matrix $C = H^T R^{-1} H$ is of rank N, i.e., it is rank deficient, and its nullspace is of dimension 1. It is straightforward to verify that the sum of the elements of all rows of C equals zero. This implies that $C\mathbf{1}_{(N+1)\times 1} = \mathbf{0}_{(N+1)\times 1}$, and hence the vector $\frac{1}{\sqrt{N}}\mathbf{1}_{(N+1)\times 1}$ is the basis of its nullspace. The solution to Eq. (6) is found by substituting

$$P_n = A_n B_n^{-1} \tag{10}$$

Note that since

$$B_n B_n^{-1} = I$$

it is

$$\begin{aligned} \frac{d}{dt}\left(B_n B_n^{-1}\right) = 0 \quad &\Rightarrow \\ \dot{B}_n B_n^{-1} + B_n \frac{d}{dt}(B_n^{-1}) = 0 \quad &\Rightarrow \\ \frac{d}{dt}(B_n^{-1}) = -B_n^{-1}\dot{B}_n B_n^{-1} & \end{aligned}$$

Substituting in Eq. (10) we have

$$\dot{P}_n = \dot{A}_n B_n^{-1} - A_n B_n^{-1}\dot{B}_n B_n^{-1} \tag{11}$$

Using Eqs. (10) and (11), Eq. (6) can be written as:

$$\dot{A}_n B_n^{-1} - A_n B_n^{-1}\dot{B}_n B_n^{-1} = Q_n - A_n B_n^{-1} C A_n B_n^{-1}$$

Multiplying both sides by B_n we have

$$\dot{A}_n - A_n B_n^{-1}\dot{B}_n = Q_n B_n - A_n B_n^{-1} C A_n$$

Separating the nonlinear from the linear terms and noting that

$$\begin{aligned} -A_n B_n^{-1}\dot{B}_n &= -A_n B_n^{-1} C A_n \Rightarrow \\ \dot{B}_n &= C A_n \end{aligned}$$

we can decompose the Riccati in the following two equations:

$$\begin{aligned} \dot{A}_n &= Q_n B_n \\ \dot{B}_n &= C A_n \end{aligned}$$

or in a matrix form

$$\begin{bmatrix} \dot{B}_n \\ \dot{A}_n \end{bmatrix} = \begin{bmatrix} 0 & C \\ Q_n & 0 \end{bmatrix} \begin{bmatrix} B_n \\ A_n \end{bmatrix} \tag{12}$$

Where the matrix

$$\mathcal{H} = \begin{bmatrix} 0 & C \\ Q_n & 0 \end{bmatrix} \tag{13}$$

is the Hamiltonian of this system. The general solution of Eq. (12) is given by

$$\begin{bmatrix} B_n(t) \\ A_n(t) \end{bmatrix} = e^{\mathcal{H}t} \begin{bmatrix} B_n(0) \\ A_n(0) \end{bmatrix} \tag{14}$$

where $A_n(0)$ and $B_n(0)$ are the initial values for these matrices. These are selected so that the identity $P_n(0) = A_n(0)B_n^{-1}(0)$ holds, i.e., $A_n(0) = P_n(0)$ and $B_n(0) = I$. Employing Taylor series expansion for computing the exponential of the Hamiltonian matrix yields:

$$\begin{aligned} e^{\mathcal{H}t} &= I + \mathcal{H}t + \frac{\mathcal{H}^2t^2}{2!} + \frac{\mathcal{H}^3t^3}{3!} + \cdots = \\ &= \begin{bmatrix} I + (CQ_n)\frac{t^2}{2!} + (CQ_n)^2\frac{t^4}{4!} + (CQ_n)^3\frac{t^6}{6!} + \cdots & C\left(\frac{t}{1!}I + (CQ_n)\frac{t^3}{3!} + (CQ_n)^2\frac{t^5}{5!} + \cdots\right) \\ Q_n\left(\frac{t}{1!}I + (CQ_n)\frac{t^3}{3!} + (CQ_n)^2\frac{t^5}{5!} + \cdots\right) & I + (CQ_n)\frac{t^2}{2!} + (CQ_n)^2\frac{t^4}{4!} + (CQ_n)^3\frac{t^6}{6!} + \cdots \end{bmatrix} \end{aligned}$$

In order to derive a simpler expression for the last relation, the eigenvalue decomposition of matrix CQ_n is employed. By simply carrying out the matrix multiplications, it is straightforward to show that this decomposition can be written as

$$CQ_n = U\Lambda_o U^{-1} = \begin{bmatrix} 1 & \mathbf{0}_{1\times N} \\ -\frac{1}{\rho^2}\mathbf{r} & I_{N\times N} \end{bmatrix} \begin{bmatrix} \rho^2 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times N} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}_{1\times N} \\ \frac{1}{\rho^2}\mathbf{r} & I_{N\times N} \end{bmatrix} \tag{15}$$

where U is the modal matrix of CQ_n and Λ_o is the diagonal matrix of eigenvalues of CQ_n. Noting that $CQ_n = (Q_nC)^T$, it becomes clear that the eigendecomposition of Q_nC can be written as $Q_nC = U^{-T}\Lambda_o U^T$. Substituting the expressions for CQ_n and Q_nC into Eq. (16) yields

$$\begin{aligned} e^{\mathcal{H}t} &= \begin{bmatrix} I + (CQ_n)\frac{t^2}{2!} + (CQ_n)^2\frac{t^4}{4!} + (CQ_n)^3\frac{t^6}{6!} + \cdots & C\left(\frac{t}{1!}I + (Q_nC)\frac{t^3}{3!} + (Q_nC)^2\frac{t^5}{5!} + \cdots\right) \\ Q_n\left(\frac{t}{1!}I + (CQ_n)\frac{t^3}{3!} + (CQ_n)^2\frac{t^5}{5!} + \cdots\right) & I + (Q_nC)\frac{t^2}{2!} + (Q_nC)^2\frac{t^4}{4!} + (Q_nC)^3\frac{t^6}{6!} + \cdots \end{bmatrix} \\ &= \begin{bmatrix} U\left(I + \Lambda_o\frac{t^2}{2!} + \Lambda_o^2\frac{t^4}{4!} + \Lambda_o^3\frac{t^6}{6!} + \cdots\right)U^{-1} & CU^{-T}\left(\frac{t}{1!}I + \Lambda_o\frac{t^3}{3!} + \Lambda_o^2\frac{t^5}{5!} + \cdots\right)U^T \\ Q_nU\left(\frac{t}{1!}I + \Lambda_o\frac{t^3}{3!} + \Lambda_o^2\frac{t^5}{5!} + \cdots\right)U^{-1} & U^{-T}\left(I + \Lambda_o\frac{t^2}{2!} + \Lambda_o^2\frac{t^4}{4!} + \Lambda_o^3\frac{t^6}{6!} + \cdots\right)U^T \end{bmatrix} \end{aligned}$$

In the above expression the following time varying terms appear:

$$\begin{aligned} K_1(t) &= I + \Lambda_o\frac{t^2}{2!} + \Lambda_o^2\frac{t^4}{4!} + \Lambda_o^3\frac{t^6}{6!} + \cdots \\ &= \begin{bmatrix} 1 + \rho^2\frac{t^2}{2!} + \rho^4\frac{t^4}{4!} + \rho^6\frac{t^6}{6!} + \cdots & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & I_{N\times N} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{\rho t}+e^{-\rho t}}{2} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & I_{N\times N} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} K_2(t) &= \frac{t}{1!}I + \Lambda_o\frac{t^3}{3!} + \Lambda_o^2\frac{t^5}{5!} + \cdots \\ &= \begin{bmatrix} \frac{t}{1!} + \rho^2\frac{t^3}{3!} + \rho^4\frac{t^5}{5!} + \cdots & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & t\, I_{N\times N} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\rho}\left(\rho\frac{t}{1!} + \rho^3\frac{t^3}{3!} + \rho^5\frac{t^5}{5!} + \cdots\right) & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & t\, I_{N\times N} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\rho}\frac{e^{\rho t}-e^{-\rho t}}{2} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & t\, I_{N\times N} \end{bmatrix} \end{aligned}$$

In order to derive simpler expressions for the above terms, the identities of Appendix A have been employed. The following expression for $e^{\mathcal{H}t}$ is therefore obtained:

$$
\begin{aligned}
e^{\mathcal{H}t} &= \begin{bmatrix} U \begin{bmatrix} \frac{e^{\rho t}+e^{-\rho t}}{2} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & I_{N\times N} \end{bmatrix} U^{-1} & CU^{-T} \begin{bmatrix} \frac{1}{\rho}\frac{e^{\rho t}-e^{-\rho t}}{2} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & t\, I_{N\times N} \end{bmatrix} U^{T} \\ Q_n U \begin{bmatrix} \frac{1}{\rho}\frac{e^{\rho t}-e^{-\rho t}}{2} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & t\, I_{N\times N} \end{bmatrix} U^{-1} & U^{-T} \begin{bmatrix} \frac{e^{\rho t}+e^{-\rho t}}{2} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & I_{N\times N} \end{bmatrix} U^{T} \end{bmatrix} \\
&= \begin{bmatrix} UK(t)U^{-1} & CL(t)U^{T} \\ Q_n UL(t)U^{-1} & U^{-T}K(t)U^{T} \end{bmatrix} \qquad (16)
\end{aligned}
$$

where we have used the notation

$$K(t) = \begin{bmatrix} \frac{e^{\rho t}+e^{-\rho t}}{2} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & I_{N\times N} \end{bmatrix}$$

and

$$L(t) = \begin{bmatrix} \frac{1}{\rho}\frac{e^{\rho t}-e^{-\rho t}}{2} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & t\, I_{N\times N} \end{bmatrix}$$

Substituting for $e^{\mathcal{H}t}$ in Eq. (14) and using the initial values $A_n(0) = P_n(0)$, $B_n(0) = I$, yields:

$$\begin{bmatrix} B_n(t) \\ A_n(t) \end{bmatrix} = e^{\mathcal{H}t} \begin{bmatrix} I \\ P_n(0) \end{bmatrix} = \begin{bmatrix} UK(t)U^{-1} + CU^{-T}L(t)U^{T}P_n(0) \\ Q_n UL(t)U^{-1} + U^{-T}K(t)U^{T}P_n(0) \end{bmatrix}$$

By carrying out the matrix multiplications, it is easy to verify that $Q_n U = U^{-T} Q_n$, and thus Eq. (17) can be written as

$$\begin{bmatrix} B_n(t) \\ A_n(t) \end{bmatrix} = \begin{bmatrix} UK(t)U^{-1} + CU^{-T}L(t)U^{T}P_n(0) \\ U^{-T}Q_n L(t)U^{-1} + U^{-T}K(t)U^{T}P_n(0) \end{bmatrix}$$

and thus the normalized covariance matrix becomes

$$
\begin{aligned}
P_n(t) &= A_n(t)B_n^{-1}(t) \\
&= \left(U^{-T}Q_n L(t)U^{-1} + U^{-T}K(t)U^{T}P_n(0)\right)\left(UK(t)U^{-1} + CU^{-T}L(t)U^{T}P_n(0)\right)^{-1} \\
&= U^{-T}\left(Q_n L(t) + K(t)U^{T}P_n(0)U\right)U^{-1}U\left(K(t) + U^{-1}CU^{-T}L(t)U^{T}P_n(0)U\right)^{-1}U^{-1} \\
&= U^{-T}\left(Q_n L(t) + K(t)P_0\right)\left(K(t) + U^{-1}CU^{-T}L(t)P_0\right)^{-1}U^{-1} \qquad (17)
\end{aligned}
$$

where we have introduced the quantity

$$P_0 = U^{T}P_n(0)U = \begin{bmatrix} P_{rr} - \frac{2}{\rho^2}\mathbf{r}^T P_{Lr} + \frac{1}{\rho^4}\mathbf{r}^T P_{LL}\mathbf{r} & P_{rL} - \frac{1}{\rho^2}\mathbf{r}^T P_{LL} \\ P_{Lr} - \frac{1}{\rho^2}P_{LL}\mathbf{r} & P_{LL} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad (18)$$

In order to facilitate the derivation of the limit of the covariance we write Eq. (17) in the form

$$
\begin{aligned}
P_n(t) &= U^{-T}\left(Q_n L(t)K^{-1}(t) + M(t)\right)U^{-1} \\
&= U^{-T}Q_n L(t)K^{-1}(t)U^{-1} + U^{-T}M(t)U^{-1} \\
&= P_a(t) + P_b(t) \qquad (19)
\end{aligned}
$$

where $M(t)$ is a matrix to be determined. We note that

$$
\begin{aligned}
P_n(t) &= U^{-T}\left(Q_n L(t)K^{-1}(t) + M(t)\right)U^{-1} \Rightarrow \\
U^{-T}\left(Q_n L(t)K^{-1}(t) + M(t)\right)U^{-1} &= U^{-T}\left(Q_n L(t) + K(t)P_0\right)\left(K(t) + U^{-1}CU^{-T}L(t)P_0\right)^{-1}U^{-1} \Rightarrow \\
Q_n L(t)K^{-1}(t) + M(t) &= \left(Q_n L(t) + K(t)P_0\right)\left(K(t) + U^{-1}CU^{-T}L(t)P_0\right)^{-1}
\end{aligned}
$$

Multiplying both sides by $\left(K(t) + U^{-1}CU^{-T}L(t)P_0\right)$ yields

$$\begin{aligned}(Q_nL(t)K^{-1}(t) + M(t))(K(t) + U^{-1}CU^{-T}L(t)P_0) &= Q_nL(t)(t) + KP_0 \Rightarrow \\ Q_nL(t) + Q_nL(t)K^{-1}(t)U^{-1}CU^{-T}L(t)P_0 + M(t)(K(t) + U^{-1}CU^{-T}L(t)P_0) &= Q_nL(t) + K(t)P_0\end{aligned}$$

And solving for $M(t)$ yields

$$M(t) \;=\; (K(t) - Q_nL(t)K^{-1}(t)U^{-1}CU^{-T}L(t))P_0(K(t) + U^{-1}CU^{-T}L(t)P_0)^{-1}$$

Evaluation of the expression $K(t) - Q_nL(t)K^{-1}(t)U^{-1}CU^{-T}L(t)$ verifies that

$$K(t) - Q_nL(t)K^{-1}(t)U^{-1}CU^{-T}L(t) = K^{-1}(t)$$

and thus

$$M(t) = K^{-1}(t)P_0(K(t) + U^{-1}CU^{-T}L(t)P_0)^{-1}$$

1.2 Steady State Covariance

It is of interest to evaluate the steady state value of this covariance matrix, i.e., the value as $t \to \infty$. We observe (cf. Eq. (19)) that the normalized covariance matrix comprises of two terms. The first term is independent of the initial conditions, and its limit value is

$$\begin{aligned}\lim_{t\to\infty} P_a(t) &= \lim_{t\to\infty} U^{-T}Q_nL(t)K^{-1}(t)U^{-1} \qquad (20)\\ &= U^{-T}\lim_{t\to\infty}\begin{bmatrix}\frac{1}{\rho}\frac{e^{\rho t}-e^{-\rho t}}{e^{\rho t}+e^{-\rho t}} & \mathbf{0}_{1\times N}\\ \mathbf{0}_{1\times N} & \mathbf{0}_{N\times N}\end{bmatrix}U^{-1}\\ &= U^{-T}\begin{bmatrix}\frac{1}{\rho} & \mathbf{0}_{1\times N}\\ \mathbf{0}_{1\times N} & \mathbf{0}_{N\times N}\end{bmatrix}U^{-1}\\ &= \begin{bmatrix}\frac{1}{\rho} & \mathbf{0}_{1\times N}\\ \mathbf{0}_{1\times N} & \mathbf{0}_{N\times N}\end{bmatrix} \qquad (21)\end{aligned}$$

The second term in Eq. (19) depends on the initial covariance matrix, and is equal to

$$\begin{aligned}\lim_{t\to\infty} P_b(t) &= \lim_{t\to\infty} U^{-T}M(t)U^{-1}\\ &= U^{-T}\left(\lim_{t\to\infty} M(t)\right)U^{-1}\end{aligned}$$

where

$$\begin{aligned}\lim_{t\to\infty} M(t) &= \lim_{t\to\infty} K^{-1}(t)P_0(K(t) + U^{-1}CU^{-T}L(t)P_0)^{-1}\\ &= \lim_{t\to\infty} K^{-1}(t)P_0\left(I + K^{-1}(t)U^{-1}CU^{-T}L(t)P_0\right)^{-1}K^{-1}(t)\end{aligned}$$

$$\lim_{t\to\infty} K^{-1}(t)\Xi(t)K^{-1}(t)$$

where

$$\Xi(t) \;=\; P_0\left(I + K^{-1}(t)U^{-1}CU^{-T}L(t)P_0\right)^{-1} \qquad (22)$$

But we note that

$$\begin{aligned}\lim_{t\to\infty} K^{-1} &= \lim_{t\to\infty}\begin{bmatrix}\frac{2}{e^{\rho t}+e^{-\rho t}} & \mathbf{0}_{1\times N}\\ \mathbf{0}_{1\times N} & I_{N\times N}\end{bmatrix}\\ &= \begin{bmatrix}0 & \mathbf{0}_{1\times N}\\ \mathbf{0}_{1\times N} & I_{N\times N}\end{bmatrix}\end{aligned}$$

Therefore we can write

$$\lim_{t\to\infty} M(t) \;=\; \begin{bmatrix}0 & \mathbf{0}_{1\times N}\\ \mathbf{0}_{1\times N} & I_{N\times N}\end{bmatrix}\left(\lim_{t\to\infty}\Xi(t)\right)\begin{bmatrix}0 & \mathbf{0}_{1\times N}\\ \mathbf{0}_{1\times N} & I_{N\times N}\end{bmatrix} \qquad (23)$$

From this expression we conclude that in order to find the steady state value of the term $P_b(t)$, only the limit of the (2,2) submatrix element of $\Xi(t)$ is necessary, since all the other submatrix elements are multiplied by zero coefficients. To this end, we first evaluate the matrix $K^{-1}(t)U^{-1}CU^{-T}L(t)$. By simply carrying out the matrix multiplications it is straightforward to show that

$$K^{-1}(t)U^{-1}CU^{-T}L(t) = \begin{bmatrix} \alpha(t)\rho & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & tA \end{bmatrix} \tag{24}$$

where

$$\alpha(t) = \frac{e^{\rho t} - e^{-\rho t}}{e^{\rho t} + e^{-\rho t}}$$

and A is a $N \times N$ constant matrix, given by

$$A = R_n^{-1} - \frac{1}{\rho^2}\mathbf{r}\mathbf{r}^T$$

We note that A is the Schur complement of ρ^2 in C, and since $\rho^2 > 0$, the rank of A is determined by the following property of Schur complements:

$$\text{rank}(C) = \text{rank}(A) + \text{rank}(\rho^2) \Rightarrow \text{rank}(A) = N - 1$$

Thus A is rank deficient, and

$$\begin{aligned} A\mathbf{1}_{N\times 1} &= \left(R_n^{-1} - \frac{1}{\rho^2}\mathbf{r}\mathbf{r}^T\right)\mathbf{1}_{N\times 1} \\ &= R_n^{-1}\mathbf{1}_{N\times 1} - \frac{1}{\rho^2}\mathbf{r}\mathbf{r}^T\mathbf{1}_{N\times 1} \\ &= \mathbf{r} - \frac{1}{\rho^2}\mathbf{r}\rho^2 \\ &= 0 \end{aligned}$$

Thus we conclude that the eigenvector associated with the zero eigenvalue of A is $A_N = \frac{1}{\sqrt{N}}\mathbf{1}_{N\times 1}$. This fact will be useful in the derivations that follow.

Using the expression of Eq. (24), $\Xi(t)$ can be expressed as

$$\begin{aligned} \Xi(t) &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \left(I + \begin{bmatrix} \alpha(t)\rho & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & tA \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 1 + \alpha(t)\rho P_{11} & \alpha(t)\rho P_{12} \\ tAP_{21} & I_{N\times N} + t\,AP_{22} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \Xi_{11}(t) & \Xi_{12}(t) \\ \Xi_{21}(t) & \Xi_{22}(t) \end{bmatrix} \end{aligned} \tag{25}$$

Using the formula for the inversion of a partitioned matrix, given in Appendix D, and carrying out the matrix multiplication yields the following expression for the (2,2) submatrix element of $\Xi(t)$:

$$\Xi_{22}(t) = F(t)(AF(t)t + I_{N\times N})^{-1}$$

where $F(t)$ is the symmetric matrix defined as

$$F(t) = P_{22} - \frac{\alpha(t)\rho P_{21}P_{12}}{1 + \alpha(t)\rho P_{11}} \tag{26}$$

In order to find the limit of Ξ_{22} at steady state, we employ the Singular Value Decomposition (SVD) of matrix $AF(t)$, which is given by:

$$AF(t) = U(t)\Sigma(t)V^T(t) = [U_1(t)\;\; U_2(t)] \begin{bmatrix} \Sigma_1(t) & \mathbf{0}_{(N-p)\times p} \\ \mathbf{0}_{p\times(N-p)} & \mathbf{0}_{p\times p} \end{bmatrix} \begin{bmatrix} V_1^T(t) \\ V_2^T(t) \end{bmatrix}$$

In the above expression, $U(t)$ and $V(t)$ are the matrices of left and right singular vectors, and $\Sigma(t)$ is the matrix of singular values. Since the matrix A is rank deficient, $AF(t)$ will also be rank deficient, and will have p (at least one) singular values equal to zero. The indices 1 and 2 in the partitioning of the matrices $U(t)$ and $V(t)$ correspond to the singular vectors associated with the positive and zero singular values of $AF(t)$, respectively.

In order to compute the limit of $\Xi_{22}(t)$ after sufficient time, we apply the following lemma, whose proof is given in Appendix B:

Lemma 1.1 *If $Y(t)$ is a square matrix, whose limit $Y(\infty) = \lim_{t\to\infty} Y(t)$ exists, and whose singular value decomposition is denoted as $Y(t) = W(t)\Lambda(t)Z^T(t)$, then*

$$\lim_{t\to\infty} (Y(t)t + I)^{-1} = Z_N(\infty)(W_N(\infty)^T Z_N(\infty))^{-1} W_N(\infty)^T \tag{27}$$

In the last expression $Z_N(\infty)$ and $W_N(\infty)$ are matrices whose column vectors are the right and left singular vectors of $Y(\infty)$ associated with the zero singular values.

We thus obtain

$$\begin{aligned}\lim_{t\to\infty} \Xi_{22}(t) &= \lim_{t\to\infty} F(AF(t)t + I_{N\times N})^{-1} \\ &= F(\infty)V_{2_\infty}\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1} U_{2_\infty}^T \end{aligned} \tag{28}$$

where U_{2_∞} and V_{2_∞} are the left and right singular vectors associated with the zero singular values of the matrix

$$AF(\infty) = A\left(P_{22} - \frac{\rho P_{21}P_{12}}{1+\rho P_{11}}\right)$$

We note that the column vectors of U_{2_∞} satisfy $U_i^T AF(\infty) = \mathbf{0}_{1\times N}$ which implies that either $U_i^T A = \mathbf{0}_{1\times N}$ or $U_i^T A \in \mathrm{Null}(F(\infty))$. From the preceding analysis of the properties of matrix A, it becomes clear that the only unit vector that satisfies $U_i^T A = \mathbf{0}_{1\times N}$ is the vector $A_N = \frac{1}{\sqrt{N}}\mathbf{1}_{N\times 1}$. Therefore the matrix U_{2_∞} can be written as

$$U_{2_\infty} = [A_N \;\; U_F]$$

where U_F is a matrix having a number of columns equal to the dimension of the nullspace of $F(\infty)$.

Similarly, we observe that the column vectors of the matrix V_{2_∞} satisfy $AF(\infty)V_i = \mathbf{0}_{N\times 1}$ which implies that either $F(\infty)V_i = cA_N$, with c an arbitrary nonzero number, or $F(\infty)V_i = \mathbf{0}_{N\times 1}$. We therefore write the matrix V_{2_∞} as

$$V_{2_\infty} = [V_A \;\; V_F]$$

where the q columns of V_A are the singular vectors that yield $F(\infty)V_i = cA_N$, and V_F are the singular vectors that form a basis of the nullspace of $F(\infty)$. We note that if A_N does not belong to the range of $F(\infty)$, then V_A does not exist in the above partitioning.

We can thus write

$$\begin{aligned}\lim_{t\to\infty} \Xi_{22}(t) &= F[V_A \;\; V_F]\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1} [A_N \;\; U_F]^T \\ &= [FV_A \;\; FV_F]\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1} [A_N \;\; U_F]^T \\ &= [A_N\mathbf{c} \;\; 0]\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1} [A_N \;\; U_F]^T \\ &= A_N\mathbf{c}\left[\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1}\right]_{11} A_N^T \\ &= \frac{1}{N}\mathbf{1}_{N\times 1}\mathbf{c}\left[\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1}\right]_{11} \mathbf{1}_{1\times N} \end{aligned} \tag{29}$$

where $\mathbf{c}$ is a $1\times q$ row vector, and $\left[\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1}\right]_{11}$ is the (1,1) submatrix of $\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1}$. From the expression in Eq. (29) we conclude that $\lim_{t\to\infty} \Xi_{22}(t)$ is a matrix of the form $m_{nn}\mathbf{1}_{N\times N}$, i.e., it is an $N\times N$ matrix having all its elements being equal. We can thus write

$$\lim_{t\to\infty} \Xi_{22}(t) = \frac{1}{N^2}\sum\left(F(\infty)V_{2_\infty}\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1} U_{2_\infty}^T\right)\mathbf{1}_{N\times N} = m_{nn}\mathbf{1}_{N\times N} \tag{30}$$

where the summation is being performed over all elements of the matrix $F(\infty)V_{2_\infty}\left(U_{2_\infty}^T V_{2_\infty}\right)^{-1} U_{2_\infty}^T$. Substitution of this expression in Eq. (23) yields

$$\lim_{t\to\infty} M(t) = \begin{bmatrix} 0 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & m_{nn}\mathbf{1}_{N\times N} \end{bmatrix}$$

and thus

$$\begin{aligned} \lim_{t\to\infty} P_b(t) &= U^{-T} \begin{bmatrix} 0 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times N} & m_{nn}\mathbf{1}_{N\times N} \end{bmatrix} U^{-1} \\ &= \begin{bmatrix} m_{nn} & m_{nn}\mathbf{1}_{1\times N} \\ m_{nn}\mathbf{1}_{N\times 1} & m_{nn}\mathbf{1}_{N\times N} \end{bmatrix} \end{aligned}$$

Finally, the steady state value of the normalized covariance matrix $P_n(t)$ is equal to

$$\begin{aligned} P_{n_{ss}} &= \lim_{t\to\infty} P_n(t) \\ &= \lim_{t\to\infty} \left(P_a(t) + P_b(t)\right) & (31) \\ &= \begin{bmatrix} \frac{1}{\rho} + m_{nn} & m_{nn}\mathbf{1}_{1\times N} \\ m_{nn}\mathbf{1}_{N\times 1} & m_{nn}\mathbf{1}_{N\times N} \end{bmatrix} & (32) \end{aligned}$$

Recalling that $P(t) = qP_n(t)$, we can compute the steady state uncertainty, P_{ss}, of SLAM in the one-degree of freedom case. We state this as a lemma:

Lemma 1.2 *Consider a robot performing Simultaneous Localization and Mapping (SLAM) in 1D, by continuously observing N features in the environment. If the covariance of the robot's odometric measurements is q, the covariance matrix of its exteroceptive measurements is R, and the initial covariance matrix of SLAM is equal to:*

$$P(0) = q \begin{bmatrix} P_{rr} & P_{rL} \\ P_{Lr} & P_{LL} \end{bmatrix}$$

then the steady state covariance is given by

$$P_{ss} = q \begin{bmatrix} \frac{1}{\rho} + m_{nn} & m_{nn}\mathbf{1}_{1\times N} \\ m_{nn}\mathbf{1}_{N\times 1} & m_{nn}\mathbf{1}_{N\times N} \end{bmatrix} \quad (33)$$

where

$$\rho^2 = q\mathbf{1}_{1\times N} R^{-1} \mathbf{1}_{N\times 1}$$

and

$$m_{nn} = \frac{1}{N^2} \sum \left(F(\infty) V_{2_\infty} \left(U_{2_\infty}^T V_{2_\infty}\right)^{-1} U_{2_\infty}^T \right) \quad (34)$$

with

$$F(\infty) = P_{LL} - \frac{\rho}{1 + \rho P_{rr} - \frac{2}{\rho}\mathbf{r}^T P_{Lr} + \frac{1}{\rho^3}\mathbf{r}^T P_{LL}\mathbf{r}} \left(P_{Lr} - \frac{1}{\rho^2}P_{LL}\mathbf{r}\right)\left(P_{rL} - \frac{1}{\rho^2}\mathbf{r}^T P_{LL}\right) \quad (35)$$

In these expressions $\mathbf{r} = qR^{-1}\mathbf{1}_{N\times 1}$ and U_{2_∞}, V_{2_∞} are matrices whose column vectors are the basis vectors of the left and right nullspace of $(qR^{-1} - \frac{1}{\rho^2}\mathbf{r}\mathbf{r}^T)F_1(\infty)$.

1.3 Special Case: Initially unknown landmark positions

Lemma 1.2 proves that for any initial value of the covariance matrix, the map estimates in 1D SLAM become fully correlated at steady state. The result of this lemma is important because it addresses the most general case of SLAM (in 1D). However, the resulting expression is quite cumbersome, and additionally, in practical applications, the following situation usually occurs:

- The robot's position estimate is initially uncorrelated from the landmarks' position estimates
- The landmarks' initial uncertainty is infinite (i.e. we have no prior knowledge about the map)

This scenario is modeled by selecting the following value for the initial covariance matrix:

$$P(0) = q \begin{bmatrix} p_{rr} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & p_{LL} I_{N\times N} \end{bmatrix} \tag{36}$$

By taking the limit of the expression for the steady state covariance as $p_{LL} \to \infty$, we derive a much simpler analytical solution for the steady state covariance in SLAM.

When the initial uncertainty is given by Eq. (36), matrix $F(\infty)$ becomes

$$\begin{aligned} F(\infty) &= p_{LL} I_{N\times N} + \frac{p_{LL}^2 \mathbf{r}\mathbf{r}^T}{\rho^3 + \rho^4 p_{rr} + p_{LL}\mathbf{r}^T\mathbf{r}} \\ &= \left(\frac{1}{p_{LL}} I_{N\times N} + \frac{1}{\rho^3(1+\rho p_{rr})} \mathbf{r}\mathbf{r}^T \right)^{-1} \end{aligned} \tag{37}$$

This matrix is nonsingular, and therefore $AF(\infty)$ has a single zero eigenvalue, caused by the zero eigenvalue of $A = (R_n^{-1} - \frac{1}{\rho^2}\mathbf{r}\mathbf{r}^T)$. Thus we can write

$$V_{2\infty} = \frac{1}{||F(\infty)^{-1}\mathbf{1}_{N\times 1}||} F(\infty)^{-1}\mathbf{1}_{N\times 1}$$

and

$$U_{2_\infty} = \frac{1}{\sqrt{N}} \mathbf{1}_{N\times 1}$$

Substitution in Eq. (34) yields

$$\begin{aligned} m_{nn} &= \frac{1}{N^2} \sum \left(F(\infty) V_{2_\infty} \left(U_{2_\infty}^T V_{2_\infty} \right)^{-1} U_{2_\infty}^T \right) \\ &= \frac{1}{N^2} \sum \left(F(\infty) \frac{1}{||F(\infty)^{-1}\mathbf{1}_{N\times 1}||} F(\infty)^{-1}\mathbf{1}_{N\times 1} \left(\frac{1}{\sqrt{N}} \mathbf{1}_{1\times N} \frac{1}{||F(\infty)^{-1}\mathbf{1}_{N\times 1}||} F(\infty)^{-1}\mathbf{1}_{N\times 1} \right)^{-1} \frac{1}{\sqrt{N}} \mathbf{1}_{1\times N} \right) \\ &= \frac{1}{N^2} \sum \left(\mathbf{1}_{N\times 1} \left(\mathbf{1}_{1\times N} F(\infty)^{-1} \mathbf{1}_{N\times 1} \right)^{-1} \mathbf{1}_{1\times N} \right) \\ &= \frac{1}{N^2 \mathbf{1}_{1\times N} F(\infty)^{-1} \mathbf{1}_{N\times 1}} \sum \mathbf{1}_{N\times 1}\mathbf{1}_{1\times N} \\ &= \frac{1}{\mathbf{1}_{1\times N} F(\infty)^{-1} \mathbf{1}_{N\times 1}} \\ &= \frac{1}{\sum F(\infty)^{-1}} \end{aligned}$$

where the sum is performed over all elements of $F(\infty)^{-1}$. Thus, employing the result of Eq. (37) we obtain

$$\begin{aligned} \sum F(\infty)^{-1} &= \sum \frac{1}{p_{LL}} I_{N\times N} + \sum \frac{\mathbf{r}\mathbf{r}^T}{\rho^3(1+\rho p_{rr})} \\ &= \frac{N}{p_{LL}} + \frac{1}{\rho^3(1+\rho p_{rr})} \sum \mathbf{r}\mathbf{r}^T \\ &= \frac{N}{p_{LL}} + \frac{1}{\rho^3(1+\rho p_{rr})} \rho^2 \sum_{i=1}^{N} \mathbf{r}_i \\ &= \frac{N}{p_{LL}} + \frac{1}{\rho^3(1+\rho p_{rr})} \rho^4 \\ &= \frac{N}{p_{LL}} + \frac{\rho}{1+\rho p_{rr}} \end{aligned}$$

We thus see that

$$
\begin{aligned}
m_{nn} &= \frac{1}{\sum F(\infty)^{-1}} \\
&= \frac{1}{\frac{N}{p_{LL}} + \frac{\rho}{1+\rho p_{rr}}} \\
&= \frac{p_{LL}(1+\rho p_{rr})}{N(1+\rho p_{rr}) + \rho p_{LL}}
\end{aligned} \tag{38}
$$

Furthermore, if the initial uncertainty of the landmark's positions is infinite, we have that

$$
\begin{aligned}
\lim_{p_{LL}\to\infty} m_{nn} &= \lim_{p_{LL}\to\infty} \frac{p_{LL}(1+\rho p_{rr})}{N(1+\rho p_{rr}) + \rho p_{LL}} \\
&= \frac{1}{\rho} + p_{rr}
\end{aligned} \tag{39}
$$

Substitution in Eq. (33) yields the steady state uncertainty of SLAM:

$$
P_{ss} = \begin{bmatrix} \frac{q}{\rho} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times N} \end{bmatrix} + \left(\frac{q}{\rho} + q p_{rr}\right) \mathbf{1}_{(N+1)\times(N+1)} \tag{40}
$$

If in addition the initial uncertainty about the robot's position is zero, as is usually the case in SLAM, the steady state covariance reduces to

$$
P_{ss} = \begin{bmatrix} \frac{q}{\rho} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times N} \end{bmatrix} + \frac{q}{\rho} \mathbf{1}_{(N+1)\times(N+1)} \tag{41}
$$

i.e., the steady state covariance of the robot is exactly double than that of the landmarks.

In order to gain more insight on the effect of the accuracy of the sensors on the steady state covariance, we consider the special case where the all the landmarks are being measured with equal accuracy. In this case, $\rho^2 = \frac{Nq}{r}$, where r is the covariance of the measurements of the landmark's positions, and the steady state covariance becomes

$$
P_{ss} = \begin{bmatrix} \sqrt{\frac{qr}{N}} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times N} \end{bmatrix} + \sqrt{\frac{qr}{N}} \mathbf{1}_{(N+1)\times(N+1)} \tag{42}
$$

In this case, we observe that the steady state accuracy depends on the *geometric mean* of the accuracy of the proprioceptive and exteroceptive sensors of the robot.

2 Simultaneous Localization and Mapping in 2-D

We consider a mobile robot moving on a planar surface, while observing N landmarks in the environment. The robot uses proprioceptive measurements (e.g., from an odometric or inertial sensor) to propagate its state estimates and exteroceptive measurements (e.g., from a laser range finder) to measure the relative positions of the map features with respect to itself. These measurements are fused using an Extended Kalman Filter (EKF) in order to produce estimates of the position of the robot and the landmarks. In our formulation, it is assumed that an upper bound for the variance of the errors in the robot's orientation estimates can be determined a priori. This allows us to decouple the task of position estimation from that of orientation estimation and facilitates the derivation of a closed-form expression for an upper bound on the positioning uncertainty.

The robot's orientation uncertainty is bounded when, for example, absolute orientation measurements from a compass or a sun sensor are available, or when perpendicularity of the walls in an indoor environment is used to infer orientation. In cases where neither approach is possible, our analysis still holds under the condition that a conservative upper bound for the orientation uncertainty is determined by alternative means, e.g., by estimating the maximum orientation error accumulated, over a certain period of time, due to the integration of noise in the odometric measurements [3]. It should be noted that the requirement for bounded orientation error covariance is not too restrictive: In the EKF framework, the nonlinear state propagation and measurement equations are linearized around the estimates of the robot's orientation. If the errors in these estimates are allowed to increase unbounded, the linearization will unavoidably become erroneous, and the estimates will diverge. Thus, in the vast majority of practical situations, provisions are made in order to constrain the robot's orientation uncertainty within given limits.

Having determined an upper bound on the orientation uncertainty of the robot allows us to decouple the task of orientation estimation from that of position estimation. The velocity and orientation of the robot are treated as measurement inputs for propagating the robot's state estimates. This formulation facilitates the derivation of upper bounds on the steady state uncertainty of SLAM, and is presented in the following sections.

2.1 Position propagation

The continuous time kinematic equations for a non-holonomic robot moving in 2-d are

$$\dot{x}_r(t) = V(t)\cos(\phi(t)) \tag{43}$$

$$\dot{y}_r(t) = V(t)\sin(\phi(t)) \tag{44}$$

where $V(t)$ is the robot's translational velocity at time t, and $\phi(t)$ is the robot's orientation. In the Kalman filter framework, the estimates of the robot's position are propagated using the measurements of the robot's velocity, $V_m(t)$, and the estimates of the robot's orientation, $\hat{\phi}(t)$, using the following equations:

$$\begin{aligned}\dot{\hat{x}}_r &= V_m(t)\cos(\hat{\phi}(t))\\ \dot{\hat{y}}_r &= V_m(t)\sin(\hat{\phi}(t))\end{aligned}$$

Clearly, these equations are time varying and nonlinear due to the dependence on the robot's orientation. By linearizing Eqs. (43) and (44) the error propagation equation for the robot's position is readily derived:

$$\begin{bmatrix} \dot{\tilde{x}}_r \\ \dot{\tilde{y}}_r \end{bmatrix} = \begin{bmatrix} \cos(\hat{\phi}(t)) & -V_m(t)\sin(\hat{\phi}(t)) \\ \sin(\hat{\phi}(t)) & V_m(t)\cos(\hat{\phi}(t)) \end{bmatrix} \begin{bmatrix} w_V(t) \\ \widetilde{\phi}(t) \end{bmatrix}$$

$$\Leftrightarrow \dot{\widetilde{X}}_r = \mathbf{0}_{2\times 2}\widetilde{X}_r + G_r(t)W(t) \tag{45}$$

where $w_V(t)$ is a white Gaussian noise sequence of variance σ_V^2, affecting the velocity measurements, and $\widetilde{\phi}(t)$ is the error in the robot's orientation estimate at time t. This is modeled as a white Gaussian noise sequence of variance σ_ϕ^2.

From Eq. (45) we deduce that the covariance matrix of the system noise affecting the robot's state is

$$\begin{aligned} Q_r(t) &= E\{G_r(t)W(t)W^T(t)G_r^T(t)\} \\ &= G_r(t)E\{W(t)W^T(t)\}G_r^T(t) \\ &= \begin{bmatrix} \cos(\hat{\phi}(t)) & -V_m(t)\sin(\hat{\phi}(t)) \\ \sin(\hat{\phi}(t)) & V_m(t)\cos(\hat{\phi}(t)) \end{bmatrix} \begin{bmatrix} \sigma_V^2 & 0 \\ 0 & \sigma_\phi^2 \end{bmatrix} \begin{bmatrix} \cos(\hat{\phi}(t)) & -V_m(t)\sin(\hat{\phi}(t)) \\ \sin(\hat{\phi}(t)) & V_m(t)\cos(\hat{\phi}(t)) \end{bmatrix}^T \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} \cos(\hat{\phi}(t)) & -\sin(\hat{\phi}(t)) \\ \sin(\hat{\phi}(t)) & \cos(\hat{\phi}(t)) \end{bmatrix} \begin{bmatrix} \sigma_V^2 & 0 \\ 0 & V_m^2(t)\sigma_\phi^2 \end{bmatrix} \begin{bmatrix} \cos(\hat{\phi}(t)) & -\sin(\hat{\phi}(t)) \\ \sin(\hat{\phi}(t)) & \cos(\hat{\phi}(t)) \end{bmatrix}^T \\
&= C(\hat{\phi}(t)) \begin{bmatrix} \sigma_V^2 & 0 \\ 0 & V_m^2(t)\sigma_\phi^2 \end{bmatrix} C^T(\hat{\phi}(t)) \qquad (46)
\end{aligned}
$$

where $C(\hat{\phi})$ denotes the rotation matrix associated with $\hat{\phi}$. The landmarks are modeled as static points in 2D space, and therefore the state propagation equations are

$$\dot{X}_{L_i}(t) = \mathbf{0}_{2\times 1}, \;\; \text{for} \;\; i = 1..N$$

Hence, the estimates for the landmark positions are propagated using the relations

$$\dot{\hat{X}}_{L_i} = \mathbf{0}_{2\times 1}, \;\; \text{for} \;\; i = 1..N$$

while the errors are propagated by

$$\dot{\widetilde{X}}_{Li} = \mathbf{0}_{2\times 1}, \;\; \text{for} \;\; i = 1..N$$

Using these results we can now write the error propagation equations for the entire system:

$$
\begin{aligned}
\dot{\widetilde{X}}(t) &= \mathbf{0}_{(2N+2)\times(2N+2)} \widetilde{X}(t) + \begin{bmatrix} G_r(t) & \mathbf{0}_{2\times 2N} \end{bmatrix} \begin{bmatrix} w_V(t) \\ \widetilde{\phi}(t) \end{bmatrix} \qquad (47) \\
\Leftrightarrow \dot{\widetilde{X}}(t) &= F(t) \widetilde{X}(t) + G(t) W(t) \qquad (48)
\end{aligned}
$$

where the state vector of the entire system has been defined as the stacked vector comprising of the position of the robot and landmarks, i.e.,

$$X(t) = \begin{bmatrix} X_r(t) \\ X_{L_1}(t) \\ \vdots \\ X_{L_N}(t) \end{bmatrix} = \begin{bmatrix} x_r(t) \\ y_r(t) \\ x_{L_1}(t) \\ y_{L_1}(t) \\ \vdots \\ x_{L_N}(t) \\ y_{L_N}(t) \end{bmatrix}$$

The covariance matrix of the system noise is given by

$$
\begin{aligned}
\mathbf{Q}(t) &= E\{G(t)W(t)W^T(t)G^T(t)\} \\
&= \begin{bmatrix} E\{G(t)W(t)W^T(t)G^T(t)\} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} \\
&= \begin{bmatrix} Q_r(t) & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} \qquad (49)
\end{aligned}
$$

2.2 Measurement Model

At every time instant, the robot measures the relative positions of each of the N landmarks in the environment. The relative position measurement associated with the ith landmark is given by:

$$z_i(t) = C^T(\phi_i(t)) \left(X_{L_i}(t) - X_r(t)\right) + n_{z_i}(t) \qquad (50)$$

where n_{z_i} is the noise affecting this measurement. By linearizing Eq. (50), the measurement error equation is obtained:

$$
\begin{aligned}
\widetilde{z}_i(t) &= z_i(t) - \hat{z}_i(t) \\
&= C^T(\hat{\phi}(t)) \left(X_{L_i}(t) - X_r(t)\right) - C^T(\hat{\phi}(t)) J \left(\hat{X}_{L_i}(t) - \hat{X}_r(t)\right) \widetilde{\phi}(t) + n_{z_i}(t)
\end{aligned}
$$

$$
\begin{aligned}
&= C^T(\hat{\phi}(t)) \begin{bmatrix} -I_{2\times 2} & .. & I_{2\times 2} & .. & \mathbf{0}_{2\times 2} \end{bmatrix} \begin{bmatrix} \widetilde{X}_r(t) \\ \widetilde{X}_{L_1}(t) \\ .. \\ \widetilde{X}_{L_i}(t) \\ .. \\ \widetilde{X}_{L_N}(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} I_{2\times 2} & -C^T(\hat{\phi}(t)) J \widehat{\Delta p}_i(t) \end{bmatrix} \begin{bmatrix} n_{z_i}(t) \\ \widetilde{\phi}(t) \end{bmatrix} \\
&= H_i(t)\widetilde{X}(t) + \Gamma_i(t) n_i(t) \qquad (51)
\end{aligned}
$$

where

$$
J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \widehat{\Delta p}_i(t) = \hat{X}_{L_i}(t) - \hat{X}_r(t)
$$

and we note that the measurement matrix for this relative position measurement can be written as

$$
H_i(t) = C^T(\hat{\phi}(t)) \begin{bmatrix} -I_{2\times 2} & .. & \underbrace{I_{2\times 2}}_{i+1 \text{ block}} & .. & \mathbf{0}_{2\times 2} \end{bmatrix} = C^T(\hat{\phi}(t)) H_{o_i} \qquad (52)
$$

where

$$
H_{o_i} = [-1 \;\; .. \;\; \underbrace{1}_{i+1} \;\; ..0] \otimes I_{2\times 2}
$$

and $\otimes$ denotes the Kronecker matrix product. Since the robot measures each of the N landmarks at each time step, the measurement matrix $\mathbf{H}(t)$ for the system is a block matrix whose block rows are $H_i(t)$, and we can write

$$
\mathbf{H}(t) = \begin{bmatrix} C^T(\hat{\phi}(t)) H_{o_1} \\ C^T(\hat{\phi}(t)) H_{o_2} \\ \vdots \\ C^T(\hat{\phi}(t)) H_{o_N} \end{bmatrix} = \mathbf{D}_{\hat{\phi}}(t)^T \mathbf{H}_o \qquad (53)
$$

where

$$
\mathbf{D}_{\hat{\phi}}(t) = I_{N\times N} \otimes C(\hat{\phi}(t))
$$

and

$$
\mathbf{H}_o = \begin{bmatrix} -\mathbf{1}_{N\times 1} & I_{2\times 2} \end{bmatrix} \otimes I_{2\times 2} \qquad (54)
$$

The covariance for the measurement error is given by

$$
\begin{aligned}
R_{ii}(t) &= \Gamma_i(t) E\{n_i(t) n_i^T(t)\} \Gamma_i^T(t) \\
&= R_{z_i}(t) + R_{\tilde{\phi}_i}(t) \qquad (55)
\end{aligned}
$$

This expression encapsulates all sources of noise and uncertainty that contribute to the measurement error $\widetilde{z}_i(t)$. More specifically, $R_{z_i}(t)$ is the covariance of the noise $n_i(t)$ in the recorded relative position measurement $z_i(t)$ and $R_i(t)$ is the additional covariance term due to the error $\widetilde{\phi}(t)$ in the orientation estimate the robot. This is given by:

$$
\begin{aligned}
R_{\tilde{\phi}_i}(t) &= C^T(\hat{\phi}(t)) J \widehat{\Delta p}_i(t) E\{\tilde{\phi}^2\} \widehat{\Delta p}_i^T(t) J^T C(\hat{\phi}(t)) \\
&= \sigma_\phi^2 C^T(\hat{\phi}(t)) J \widehat{\Delta p}_i(t) \widehat{\Delta p}_i^T(t) J^T C(\hat{\phi}(t)) \qquad (56)
\end{aligned}
$$

From this expression we conclude that the uncertainty σ_ϕ^2 in the orientation estimate $\hat{\phi}(t)$ of the robot is amplified by the distance between the robot and corresponding landmark.

The measurement of the relative position of the ith landmark is comprised of the distance ρ_i and bearing θ_i to this landmark, expressed in the robot's local coordinate frame, i.e.,

$$z_i(t) = \begin{bmatrix} \rho_i(t)\cos\theta_i(t) \\ \rho_i(t)\sin\theta_i(t) \end{bmatrix} + n_{z_i}(t)$$

By linearizing, the noise in this measurement can be expressed as:

$$n_{z_i}(t) \simeq \begin{bmatrix} \cos\hat{\theta}_i(t) & -\hat{\rho}_i(t)\sin\hat{\theta}_i(t) \\ \sin\hat{\theta}_i(t) & \hat{\rho}_i(t)\cos\hat{\theta}_i(t) \end{bmatrix} \begin{bmatrix} n_{\rho_i}(t) \\ n_{\theta_i}(t) \end{bmatrix}$$

where n_{ρ_i} is the error in the range measurement, n_{θ_i} is the error in the bearing measurement, and

$$\begin{aligned} \hat{\rho}_i^2(t) &= \widehat{\Delta p}_i^T(t)\widehat{\Delta p}_i(t) \\ \hat{\theta}_i(t) &= \mathrm{Atan2}(\widehat{\Delta y}_i(t), \widehat{\Delta x}_i(t)) - \hat{\phi}(t) \end{aligned}$$

are the estimates of the range and bearing to the landmark, expressed with respect to the robot's coordinate frame. At this point we note that

$$\begin{aligned} C(\hat{\phi}(t))n_{z_i}(t) &= \begin{bmatrix} \cos\hat{\phi}(t) & -\sin\hat{\phi}(t) \\ \sin\hat{\phi}(t) & \cos\hat{\phi}(t) \end{bmatrix} \begin{bmatrix} \cos\hat{\theta}_i(t) & -\hat{\rho}_i(t)\sin\hat{\theta}_i(t) \\ \sin\hat{\theta}_i(t) & \hat{\rho}_i(t)\cos\hat{\theta}_i(t) \end{bmatrix} \begin{bmatrix} n_{\rho_i}(t) \\ n_{\theta_i}(t) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\hat{\phi}(t)+\hat{\theta}_i(t)) & -\hat{\rho}_i(t)\sin(\hat{\phi}(t)+\hat{\theta}_i(t)) \\ \sin(\hat{\phi}(t)+\hat{\theta}_i) & \hat{\rho}_i(t)\cos(\hat{\phi}(t)+\hat{\theta}_i(t)) \end{bmatrix} \begin{bmatrix} n_{\rho_i}(t) \\ n_{\theta_i}(t) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\hat{\rho}_i}\widehat{\Delta p}_i(t) & J\widehat{\Delta p}_i(t) \end{bmatrix} \begin{bmatrix} n_{\rho_i}(t) \\ n_{\theta_i}(t) \end{bmatrix} \end{aligned}$$

and therefore the quantity $R_{z_i}(t)$ can be written as:

$$\begin{aligned} R_{z_i}(t) &= E\{n_{z_i}(t)n_{z_i}^T(t)\} \\ &= C^T(\hat{\phi}(t)) \begin{bmatrix} \frac{1}{\hat{\rho}_i(t)}\widehat{\Delta p}_i(t) & J\widehat{\Delta p}_i(t) \end{bmatrix} E\left\{ \begin{bmatrix} n_{\rho_i} \\ n_{\theta_i} \end{bmatrix} \begin{bmatrix} n_{\rho_i} \\ n_{\theta_i} \end{bmatrix}^T \right\} \begin{bmatrix} \frac{1}{\hat{\rho}_i(t)}\widehat{\Delta p}_i(t) & J\widehat{\Delta p}_i(t) \end{bmatrix}^T C(\hat{\phi}(t)) \\ &= C^T(\hat{\phi}(t)) \begin{bmatrix} \frac{1}{\hat{\rho}_i(t)}\widehat{\Delta p}_i(t) & J\widehat{\Delta p}_i(t) \end{bmatrix} \begin{bmatrix} \sigma_\rho^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\hat{\rho}_i(t)}\widehat{\Delta p}_i(t) & J\widehat{\Delta p}_i(t) \end{bmatrix}^T C(\hat{\phi}(t)) \\ &= C^T(\hat{\phi}(t)) \left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2(t)} \widehat{\Delta p}_i(t)\widehat{\Delta p}_i^T(t) + \sigma_\theta^2 J\widehat{\Delta p}_i(t)\widehat{\Delta p}_i^T(t)J^T \right) C(\hat{\phi}(t)) \\ &= C^T(\hat{\phi}(t)) \left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2(t)} \left(\hat{\rho}_i^2(t) I_{2\times 2} - J\widehat{\Delta p}_i(t)\widehat{\Delta p}_i^T(t)J^T \right) + \sigma_\theta^2 J\widehat{\Delta p}_i(t)\widehat{\Delta p}_i^T(t)J^T \right) C(\hat{\phi}(t)) \\ &= C^T(\hat{\phi}(t)) \left(\sigma_\rho^2 I_{2\times 2} + \left(\sigma_\theta^2 - \frac{\sigma_\rho^2}{\hat{\rho}_i^2(t)} \right) J\widehat{\Delta p}_i(t)\widehat{\Delta p}_i^T(t)J^T \right) C(\hat{\phi}(t)) \end{aligned} \quad (57)$$

where the variance of the noise in the distance and bearing measurements is given by

$$\sigma_\rho^2 = E\{n_{\rho_i}^2(t)\} \;\;,\;\; \sigma_\theta^2 = E\{n_{\theta_i}^2(t)\}$$

respectively. Due to the existence of the common error component attributed to $\tilde{\phi}(t)$, the measurements that each robot performs are correlated. The matrix of correlation between the errors in the measurements $z_i(t)$ and $z_j(t)$ is

$$\begin{aligned} R_{ij}(t) &= \Gamma(t)E\{n_i(t)n_j^T(t)\}\Gamma^T(t) \\ &= \sigma_\phi^2 C^T(\hat{\phi}(t))J\widehat{\Delta p}_i(t)\widehat{\Delta p}_j^T(t)J^T C(\hat{\phi}(t)) \end{aligned} \quad (58)$$

Using the results of Eqs. (56), (57), and (58), the covariance matrix of all the measurements performed by the robot can now be computed. This is a matrix whose 2×2 block diagonal elements equal $R_{ii}(t),\ i = 1\ldots N$ while its off-diagonal block elements are $R_{ij\ell}(t),\ i,j = 1\dots N, i \neq j$. It is easy to see that this matrix can be written in a more convenient form as:

$$\mathbf{R}(t) = \mathbf{D}_{\hat{\phi}}(t)^T \mathbf{R}_o(t) \mathbf{D}_{\hat{\phi}}(t) \tag{59}$$

where

$$\begin{aligned}
\mathbf{R}_o(t) &= \begin{bmatrix} \sigma_\rho^2 I_{2\times 2} + \left(\sigma_\phi^2 + \sigma_\theta^2 - \frac{\sigma_\rho^2}{\hat{\rho}_1^2(t)}\right) J\widehat{\Delta p}_1(t)\widehat{\Delta p}_1^T(t) J^T & \dots & \sigma_\phi^2 J\widehat{\Delta p}_1(t)\widehat{\Delta p}_N^T(t) J^T \\ \vdots & \ddots & \vdots \\ \sigma_\phi^2 J\widehat{\Delta p}_N(t)\widehat{\Delta p}_1^T(t) J^T & \dots & \sigma_\rho^2 I_{2\times 2} + \left(\sigma_\phi^2 + \sigma_\theta^2 - \frac{\sigma_\rho^2}{\hat{\rho}_N^2(t)}\right) J\widehat{\Delta p}_N(t)\widehat{\Delta p}_N^T(t) J^T \end{bmatrix} \\
&= \sigma_\rho^2 I_{2N\times 2N} + D(t)\left(\sigma_\theta^2 I_{N\times N} + \sigma_{\phi_i}^2 \mathbf{1}_{N\times N} - \operatorname{diag}\left(\frac{\sigma_{\rho_i}^2}{\rho_i^2(t)}\right)\right) D^T(t) \\
&= \underbrace{\sigma_\rho^2 I_{2N\times 2N} - D(t)\operatorname{diag}\left(\frac{\sigma_{\rho_i}^2}{\rho_i^2(t)}\right) D^T(t)}_{R_1(t)} + \underbrace{\sigma_{\theta_i}^2 D(t) D^T(t)}_{R_2(t)} + \underbrace{\sigma_{\phi_i}^2 D(t)\mathbf{1}_{N\times N} D^T(t)}_{R_3(t)}
\end{aligned} \tag{60}$$

and

$$D(t) = \begin{bmatrix} J\widehat{\Delta p}_1(t) & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix} = \mathbf{Diag}(J\widehat{\Delta p}_i(t))$$

is a $2N \times N$ block diagonal matrix, depending on the positions of the robot and landmarks. In the previous expression, the covariance term $R_1(t)$ is the covariance of the error due to the noise in the range measurements, $R_2(t)$ is the covariance term due to the error in the bearing measurements, and $R_3(t)$ is the covariance term due to the error in the orientation estimates of the robot. We are now able to compute the matrix $\mathbf{H}^T(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)$, that appears in the covariance update equations of the Kalman Filter. Substitution from Eqs. (53) and (59) results in the terms depending on the robot's orientation being canceled, i.e.,

$$\mathbf{H}^T(t)\mathbf{R}^{-1}(t)\mathbf{H}(t) \quad = \quad \mathbf{H}_o^T \mathbf{R}_o^{-1}(t)\mathbf{H}_o$$

It is interesting to observe that since the matrix $\mathbf{R}_o(t)$ does not depend on the orientation of the robot, the matrix $\mathbf{H}^T(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)$ depends only on the positions of the robot and landmarks.

2.3 The Riccati Differential Equation

The results presented in the preceding sections allow us to derive the Riccati differential equation that describes the time evolution of the covariance matrix in SLAM. This is

$$\begin{aligned}
\dot{\mathbf{P}}(t) &= \mathbf{Q}(t) - \mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t) \\
&= \mathbf{Q}(t) - \mathbf{P}(t)\mathbf{H}_o^T\mathbf{R}_o^{-1}(t)\mathbf{H}\mathbf{P}(t)
\end{aligned} \tag{61}$$

It becomes evident that this is a time-varying Riccati equation, and therefore no closed form solution for it can be found in the general case of the robot's motion. However, the following lemma allows us to derive an *upper bound* on the covariance of the position estimates:

Lemma 2.1 *If the matrices $\bar{\mathbf{R}}$ and $\bar{\mathbf{Q}}$ satisfy $\bar{\mathbf{R}} \succeq \mathbf{R}_o(t)$ and $\bar{\mathbf{Q}} \succeq \mathbf{Q}(t)$ for all $t > 0$, then the solution to the Riccati differential equation*

$$\dot{\bar{\mathbf{P}}}(t) \quad = \quad \bar{\mathbf{Q}} - \bar{\mathbf{P}}(t)\mathbf{H}_o^T\bar{\mathbf{R}}^{-1}\mathbf{H}_o\bar{\mathbf{P}}(t) \tag{62}$$

is an upper bound to the solution of the Riccati equation in Eq. (61), i.e., it satisfies $\bar{\mathbf{P}}(t) \succeq \mathbf{P}(t)$ for all $t > 0$, when the initial values of the two differential equations are equal.

The proof of this lemma is given in Appendix C. We now compute upper bounds on the matrices $\mathbf{Q}(t)$ and $\mathbf{R}_o(t)$, that will be used in order to formulate a constant coefficient Riccati differential equation for the upper bound on the positioning covariance.

In order to derive an upper bound for the system noise covariance matrix $\mathbf{Q}(t)$ we note that (cf. Eqs. (49) and (46))

$$\mathbf{Q}(t) = \begin{bmatrix} Q_r(t) & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} = \begin{bmatrix} C(\hat{\phi}(t)) \begin{bmatrix} \delta t^2 \sigma_V^2 & 0 \\ 0 & \delta t^2 V_m^2(t) \sigma_\phi^2 \end{bmatrix} C^T(\hat{\phi}(t)) & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix}$$

From the properties of rotation matrices it is known that $C^{-1}(\hat{\phi}(t)) = C^T(\hat{\phi}(t))$, and thus $Q_r(t)$ is related by a similarity transformation to the matrix

$$\begin{bmatrix} \delta t^2 \sigma_V^2 & 0 \\ 0 & \delta t^2 V_m^2(t) \sigma_\phi^2 \end{bmatrix}$$

which implies that the eigenvalues of $Q_r(t)$ are $\delta t^2 \sigma_V^2$ and $\delta t^2 V_m^2(t) \sigma_\phi^2$. We assume that the robot's velocity is approximately constant, and equal to V, and denote

$$\bar{q} = \max\left(\delta t^2 \sigma_V^2, \delta t^2 V_m^2(t) \sigma_\phi^2\right) \simeq \max\left(\delta t^2 \sigma_V^2, \delta t^2 V^2 \sigma_\phi^2\right) \tag{63}$$

This definition states that $\bar{q}$ is the largest eigenvalue of $Q_r(t)$, and therefore

$$Q_r(t) \preceq \bar{q} I_{2\times 2} \Rightarrow \mathbf{Q}(t) \preceq \begin{bmatrix} \bar{q} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} = \bar{\mathbf{Q}} \tag{64}$$

We next derive an upper bound for $\mathbf{R}_o$ by considering each of its terms separately: the term expressing the effect of the noise in the range measurements is

$$R_1(t) = \sigma_\rho^2 I_{2N\times 2N} - D(t)\,\mathrm{diag}\left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2}\right) D^T(t) \preceq \sigma_\rho^2 I_{2N\times 2N} \tag{65}$$

The last matrix inequality follows from the fact that the term being subtracted from $\sigma_\rho^2 I_{2N\times 2N}$ is a positive semidefinite matrix. The covariance term due to the noise in the bearing measurement is

$$\begin{aligned} R_2(t) &= \sigma_\theta^2 D(t) D^T(t) \\ &= \sigma_\theta^2 \mathbf{Diag}\left(\hat{\rho}_i^2(t) \begin{bmatrix} \sin^2(\hat{\theta}_i(t)) & \sin(\hat{\theta}_i(t))\cos(\hat{\theta}_i(t)) \\ \sin(\hat{\theta}_i(t))\cos(\hat{\theta}_i(t)) & \cos^2(\hat{\theta}_i(t)) \end{bmatrix}\right) \\ &\preceq \sigma_\theta^2 \mathbf{Diag}\left(\hat{\rho}_i^2(t) I_{2\times 2}\right) \\ &\preceq \sigma_\theta^2 \rho_o^2 I_{2N\times 2N} \end{aligned} \tag{66}$$

where ρ_o is the maximum possible distance between the robot and any landmark. Finally, the covariance term due to the error in the orientation of the robot is $R_3(t) = \sigma_{\phi_i}^2 D(t) \mathbf{1}_{N\times N} D^T(t)$. Calculation of the eigenvalues of the matrices $\mathbf{1}_{N\times N}$ and $I_{N\times N}$ verifies that $\mathbf{1}_{N\times N} \preceq N I_{N\times N}$, and thus we can write $R_3(t) \preceq N\sigma_\phi^2 D(t) D^T(t)$. By derivations analogous to those employed to yield an upper bound for $R_2(t)$, we can show that

$$R_3(t) \preceq N \sigma_\phi^2 \rho_o^2 I_{2N\times 2N}$$

By combining this result with those of Eqs. (65), (66), we can write $\mathbf{R}_o(t) = R_1(t) + R_2(t) + R_3(t) \preceq \bar{\mathbf{R}}$, where

$$\bar{\mathbf{R}} = \left(\sigma_\rho^2 + N\sigma_\phi^2 \rho_o^2 + \sigma_\theta^2 \rho_o^2\right) I_{2N\times 2N} = r I_{2N\times 2N} \tag{67}$$

with

$$r = \sigma_\rho^2 + N\sigma_\phi^2 \rho_o^2 + \sigma_\theta^2 \rho_o^2 \tag{68}$$

We can therefore formulate the following Riccati differential equation for the upper bound on the positioning accuracy of SLAM:

$$\begin{aligned} \dot{\bar{\mathbf{P}}}(t) &= \bar{\mathbf{Q}} - \bar{\mathbf{P}}(t)\mathbf{H}_o^T \bar{\mathbf{R}}^{-1} \mathbf{H}_o \bar{\mathbf{P}}(t) \\ &= q\bar{\mathbf{Q}}_n - \frac{1}{r}\bar{\mathbf{P}}(t)\mathbf{H}_o^T \mathbf{H}_o \bar{\mathbf{P}}(t) \end{aligned} \tag{69}$$

with

$$\bar{\mathbf{Q}}_n = \begin{bmatrix} I_{2\times2} & \mathbf{0}_{2\times2N} \\ \mathbf{0}_{2N\times2} & \mathbf{0}_{2N\times2N} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times1} & \mathbf{0}_{N\times N} \end{bmatrix} \otimes I_{2\times2}$$

The initial value for the covariance of SLAM is assumed to be equal to:

$$\mathbf{P}(0) \;=\; \begin{bmatrix} \mathbf{P}_{rr_o} & \mathbf{0}_{2\times2N} \\ \mathbf{0}_{2N\times2} & \mathbf{P}_{LL_o} \end{bmatrix} \tag{70}$$

in other words, we assume that the initial position estimates for the for the robot are uncorrelated from the map features' position estimates. Additionally, in order to simplify the derivations that follow, we assume that $\mathbf{P}_{LL_o}$ is an invertible matrix. These assumptions are not necessary in order to derive a solution. By following steps analogous to the 1D case, a solution for the most general case can be derived. However, as the analysis of the 1D problem demonstrates, the resulting expressions are too cumbersome. Moreover, in most practical cases the aforementioned assumptions are met, and therefore, we will employ them in the ensuing analysis, in order to provide simpler and more intuitive results.

2.4 Upper Bound on the Steady State Covariance Matrix

We now focus on deriving the asymptotic solution to the Riccati differential equation in Eq. (69), in order to characterize the steady state performance of SLAM. The analysis is simplified by introducing the *normalized* covariance matrix:

$$\mathbf{P}_n(t) \;=\; \frac{1}{\bar{q}}\bar{\mathbf{P}}(t)$$

which leads to the following Riccati differential equation:

$$\begin{aligned} \dot{\mathbf{P}}_n(t) \;&=\; \bar{\mathbf{Q}}_n - \frac{\bar{q}}{r}\mathbf{P}_n(t)\mathbf{H}_o^T\mathbf{H}_o\mathbf{P}_n(t) && (71) \\ &=\; \bar{\mathbf{Q}}_n - \mathbf{P}_n(t)\mathbf{C}\mathbf{P}_n(t) && (72) \end{aligned}$$

with initial condition

$$\mathbf{P}_n(0) = \frac{1}{\bar{q}}\mathbf{P}(0) = \frac{1}{\bar{q}}\begin{bmatrix} \mathbf{P}_{rr_o} & \mathbf{0}_{2\times2N} \\ \mathbf{0}_{2N\times2} & \mathbf{P}_{LL_o} \end{bmatrix}$$

In the last expression the matrix $\mathbf{C}$ is defined as

$$\begin{aligned} \mathbf{C} \;&=\; \frac{\bar{q}}{r}\mathbf{H}_o^T\mathbf{H}_o \\ &=\; \begin{bmatrix} \frac{N\bar{q}}{r}I_{2\times2} & -\frac{\bar{q}}{r}\mathbf{J}^T \\ -\frac{\bar{q}}{r}\mathbf{J} & \frac{\bar{q}}{r}I_{2N\times2N} \end{bmatrix} && (73) \end{aligned}$$

where

$$\mathbf{J} = \mathbf{1}_{N\times1} \otimes I_{2\times2}$$

At this point we note that the matrix $\mathbf{C}$ is singular. Analytical evaluation of its eigenvalues shows that zero is an eigenvalue of $\mathbf{C}$ with multiplicity 2, and thus $\mathrm{rank}(\mathbf{C}) = 2N$.

The solution to the Riccati equation in Eq. (71) is obtained by a derivation process analogous to that employed in the 1D case. Specifically, we note that the eigendecomposition of the matrix $\mathbf{C}\bar{\mathbf{Q}}_n$ is written as:

$$\mathbf{C}\bar{\mathbf{Q}}_n = \mathbf{U}\mathbf{\Lambda}_o\mathbf{U}^{-1} = \begin{bmatrix} I_{2\times2} & \mathbf{0}_{2\times2N} \\ -\frac{1}{N}\mathbf{J} & I_{2N\times2N} \end{bmatrix}\begin{bmatrix} \boldsymbol{\rho}^2 I_{2\times2} & \mathbf{0}_{2\times2N} \\ \mathbf{0}_{2N\times2} & \mathbf{0}_{2N\times2N} \end{bmatrix}\begin{bmatrix} I_{2\times2} & \mathbf{0}_{2\times2N} \\ \frac{1}{N}\mathbf{r} & I_{2N\times2N} \end{bmatrix} \tag{74}$$

where we have denoted

$$\boldsymbol{\rho}^2 = \frac{N\bar{q}}{r},$$

$\mathbf{U}$ is the matrix comprising of the eigenvectors of $\mathbf{C}\bar{\mathbf{Q}}_n$ as its column vectors, and $\mathbf{\Lambda}_o$ is the matrix of eigenvalues of $\mathbf{C}\bar{\mathbf{Q}}_n$. The solution to the Riccati differential equation in Eq. (71) is derived by forming the Hamiltonian matrix, and evaluating its exponential function. The derivations are analogous to the 1D case, and therefore we directly present the final form of the solution:

$$\begin{aligned}\mathbf{P}_n(t) &= \mathbf{U}^{-T}\bar{\mathbf{Q}}_n\mathbf{L}(t)\mathbf{K}^{-1}(t)\mathbf{U}^{-1} + \mathbf{U}^{-T}\mathbf{M}(t)\mathbf{U}^{-1} \\ &= \mathbf{P}_a(t) + \mathbf{P}_b(t) \end{aligned} \tag{75}$$

where we have denoted

$$\mathbf{K}(t) = \begin{bmatrix} \frac{e^{\rho t}+e^{-\rho t}}{2} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2\times 2N} & I_{2N\times 2N} \end{bmatrix}$$

and

$$\mathbf{L}(t) = \begin{bmatrix} \frac{1}{\rho}\frac{e^{\rho t}-e^{-\rho t}}{2} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2\times 2N} & t\, I_{2N\times 2N} \end{bmatrix}$$

while the matrix $\mathbf{M}(t)$ is equal to

$$\mathbf{M}(t) = \mathbf{K}^{-1}(t)\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U}(\mathbf{K}(t) + \mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-T}\mathbf{L}(t)\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U})^{-1} \tag{76}$$

In order to derive an upper bound on the steady state uncertainty of the position estimates in SLAM, we need to evaluate the asymptotic value towards which the solution in Eq. (75) converges. To this end, we evaluate the limits of the quantities $\mathbf{P}_a(t)$ and $\mathbf{P}_b(t)$ as $t \to \infty$. The derivations are once again similar to those employed in the 1D case. Specifically, the limit of the term $\mathbf{P}_a(t)$ is equal to

$$\lim_{t\to\infty} \mathbf{P}_a(t) = \lim_{t\to\infty} \mathbf{U}^{-T}\bar{\mathbf{Q}}_n\mathbf{L}(t)\mathbf{K}^{-1}(t)\mathbf{U}^{-1} \tag{77}$$

$$= \begin{bmatrix} \frac{1}{\rho} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2\times 2N} & \mathbf{0}_{2N\times 2N} \end{bmatrix} \tag{78}$$

The limit of the second term in Eq. (75) is equal to

$$\begin{aligned}\lim_{t\to\infty} \mathbf{P}_b(t) &= \lim_{t\to\infty} \mathbf{U}^{-T}\mathbf{M}(t)\mathbf{U}^{-1} \\ &= \mathbf{U}^{-T}\left(\lim_{t\to\infty}\mathbf{M}(t)\right)\mathbf{U}^{-1}\end{aligned} \tag{79}$$

where (cf. Eq. (76))

$$\begin{aligned}\lim_{t\to\infty}\mathbf{M}(t) &= \lim_{t\to\infty}\mathbf{K}^{-1}(t)\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U}\left(I_{(2N+2)\times(2N+2)} + \mathbf{K}^{-1}(t)\mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-T}\mathbf{L}(t)\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U}\right)^{-1}\mathbf{K}^{-1}(t) \\ &= \lim_{t\to\infty}\mathbf{K}^{-1}(t)\mathbf{U}^T\mathbf{P}_n(0)\left(\mathbf{U}^{-1} + \mathbf{K}^{-1}(t)\mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-T}\mathbf{L}(t)\mathbf{U}^T\mathbf{P}_n(0)\right)^{-1}\mathbf{K}^{-1} \\ &= \lim_{t\to\infty}\mathbf{K}^{-1}(t)\mathbf{U}^T\mathbf{P}_n(0)\left(I_{(2N+2)\times(2N+2)} + \mathbf{U}\mathbf{K}^{-1}(t)\mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-T}\mathbf{L}(t)\mathbf{U}^T\mathbf{P}_n(0)\right)^{-1}\mathbf{U}\mathbf{K}^{-1}(t) \\ &= \lim_{t\to\infty}\mathbf{K}^{-1}(t)\mathbf{U}^T\mathbf{\Xi}(t)\mathbf{U}\mathbf{K}^{-1}(t)\end{aligned}$$

where

$$\mathbf{\Xi}(t) = \mathbf{P}_n(0)\left(I_{(2N+2)\times(2N+2)} + \mathbf{U}\mathbf{K}^{-1}(t)\mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-T}\mathbf{L}(t)\mathbf{U}^T\mathbf{P}_n(0)\right)^{-1} \tag{80}$$

But we note that

$$\begin{aligned}\lim_{t\to\infty}\mathbf{K}^{-1}\mathbf{U}^T &= \lim_{t\to\infty}(\mathbf{U}\mathbf{K}^{-1})^T \\ &= \lim_{t\to\infty}\begin{bmatrix} \frac{2}{e^{\rho t}+e^{-\rho t}} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2\times 2N} & I_{2N\times 2N} \end{bmatrix}\begin{bmatrix} I_{2\times 2} & -\frac{1}{N}\mathbf{J}^T \\ \mathbf{0}_{2N\times 2} & I_{2N\times 2N} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2\times 2N} & I_{2N\times 2N} \end{bmatrix}\end{aligned}$$

Therefore we can write

$$\lim_{t\to\infty}\mathbf{M}(t) = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2\times 2N} & I_{2N\times 2N} \end{bmatrix}\left(\lim_{t\to\infty}\mathbf{\Xi}(t)\right)\begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2\times 2N} & I_{2N\times 2N} \end{bmatrix} \tag{81}$$

At this point we introduce the following partitioning of the matrix $\Xi(t)$:

$$\Xi(t) = \begin{bmatrix} \Xi_{11}(t) & \Xi_{12}(t) \\ \Xi_{21}(t) & \Xi_{22}(t) \end{bmatrix} = \begin{bmatrix} 2 \times 2 & 2 \times 2N \\ 2N \times 2 & 2N \times 2N \end{bmatrix}$$

And therefore

$$\lim_{t \to \infty} \mathbf{M}(t) \quad = \quad \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2\times 2N} & \Xi_{22}(t) \end{bmatrix} \tag{82}$$

From this expression we conclude that in order to find the steady state value of the term $\mathbf{P}_b(t)$, only the limit of the (2,2) submatrix element of $\Xi(t)$ is necessary.

Carrying out the matrix multiplication we obtain

$$\mathbf{U}\mathbf{K}^{-1}(t)\mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-T}\mathbf{L}(t)\mathbf{U}^{T} = \begin{bmatrix} \alpha(t)\boldsymbol{\rho} I_{2\times 2} & -\frac{\alpha(t)\boldsymbol{\rho}}{N}\mathbf{J}^T \\ -\frac{\alpha(t)\boldsymbol{\rho}}{N}\mathbf{J} & \frac{\alpha(t)\boldsymbol{\rho}}{N}\mathbf{J}\mathbf{J}^T + t\,\mathbf{A} \end{bmatrix} \tag{83}$$

where

$$\alpha(t) = \frac{e^{\boldsymbol{\rho} t} - e^{-\boldsymbol{\rho} t}}{e^{\boldsymbol{\rho} t} + e^{-\boldsymbol{\rho} t}}$$

and $\mathbf{A}$ is a $2N \times 2N$ constant matrix, given by

$$\mathbf{A} = \frac{r}{\bar{q}} I_{2N\times 2N} - \frac{r}{N\bar{q}}\mathbf{J}\mathbf{J}^T$$

Similarly to the 1D case, $\mathbf{A}$ is the Schur complement of $\boldsymbol{\rho}^2 I_{2\times 2}$ in $\mathbf{C}$, and the following property holds:

$$\mathrm{rank}(\mathbf{C}) = \mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\boldsymbol{\rho}^2 I_{2\times 2}) \Rightarrow \mathrm{rank}(\mathbf{A}) = 2N - 2$$

Thus $\mathbf{A}$ is rank deficient, and it is easy to see that $\mathbf{A}\mathbf{J} = \mathbf{0}_{2N\times 2}$, which implies that the column vectors of the matrix $\mathbf{V}_N = \frac{1}{\sqrt{N}}\mathbf{J}$ form a basis of the nullspace of $\mathbf{A}$.

Using the expression of Eq. (83), $\Xi(t)$ can be expressed as

$$\begin{aligned}
\Xi(t) &= \frac{1}{\bar{q}} \begin{bmatrix} \mathbf{P}_{rr_o} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{P}_{LL_o} \end{bmatrix} \left(I_{(2N+2)\times(2N+2)} + \frac{1}{\bar{q}} \begin{bmatrix} \alpha(t)\boldsymbol{\rho} I_{2\times 2} & -\frac{\alpha(t)\boldsymbol{\rho}}{N}\mathbf{J}^T \\ -\frac{\alpha(t)\boldsymbol{\rho}}{N}\mathbf{J} & \frac{\alpha(t)\boldsymbol{\rho}}{N}\mathbf{J}\mathbf{J}^T + t\,\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{rr_o} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{P}_{LL_o} \end{bmatrix} \right)^{-1} \\
&= \frac{1}{\bar{q}} \begin{bmatrix} \mathbf{P}_{rr_o} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{P}_{LL_o} \end{bmatrix} \begin{bmatrix} I_{2\times 2} + \frac{\alpha(t)\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o} & -\frac{\alpha(t)\boldsymbol{\rho}}{N\bar{q}}\mathbf{J}^T\mathbf{P}_{LL_o} \\ -\frac{\alpha(t)\boldsymbol{\rho}}{N\bar{q}}\mathbf{J}\mathbf{P}_{rr_o} & I_{2N\times 2N} + \frac{1}{\bar{q}}\left(\frac{\alpha(t)\boldsymbol{\rho}}{N}\mathbf{J}\mathbf{J}^T + t\,\mathbf{A}\right)\mathbf{P}_{LL_o} \end{bmatrix}^{-1} \\
&= \frac{1}{\bar{q}} \begin{bmatrix} \mathbf{P}_{rr_o} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{P}_{LL_o} \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_3 & \mathbf{P}_4 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \Xi_{11}(t) & \Xi_{12}(t) \\ \Xi_{21}(t) & \Xi_{22}(t) \end{bmatrix}
\end{aligned} \tag{84}$$

Using the formula for the inversion of a partitioned matrix, given in Appendix D, and carrying out the matrix multiplication yields the following expression for $\Xi_{22}(t)$:

$$\Xi_{22}(t) \;=\; \frac{1}{\bar{q}}\mathbf{P}_{LL_o}(\mathbf{P}_4 - \mathbf{P}_3\mathbf{P}_1^{-1}\mathbf{P}_2)^{-1}$$

Substitution of the values of matrices $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$, defined in Eq. (84), and simple algebraic manipulation yields the following expression for $\Xi_{22}(t)$:

$$\Xi_{22}(t) = \frac{1}{\bar{q}}\mathbf{P}_{LL_o}\left(\frac{1}{\bar{q}}\mathbf{A}\mathbf{P}_{LL_o}t + I_{2N\times 2N} + \frac{\alpha(t)\boldsymbol{\rho}}{N^2\bar{q}}\mathbf{J}\left(I_{2\times 2} + \frac{\alpha(t)\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1}\mathbf{J}^T\mathbf{P}_{LL_o}\right)^{-1}$$

A this point we assume that $\mathbf{P}_{LL_o}$ is invertible, in order to simplify the derivations, although this is clearly not necessary for deriving a closed form solution, as the 1D analysis demonstrates. With this assumption the preceding expression can be written as

$$\boldsymbol{\Xi}_{22}(t) = \left(\mathbf{A}t + \bar{q}\mathbf{P}_{LL_o}^{-1} + \frac{\alpha(t)\boldsymbol{\rho}}{N^2}\mathbf{J}\left(I_{2\times 2} + \frac{\alpha(t)\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1}\mathbf{J}^T \right)^{-1}$$

The steady state value of this matrix can be computed by application of Lemma 1.1. Specifically, $\mathbf{A}$ is singular and $\mathbf{V}_N = \frac{1}{\sqrt{N}}\mathbf{J}$ is the matrix whose columns comprise the basis of its nullspace. Moreover,

$$\lim_{t\to\infty}\left(\bar{q}\mathbf{P}_{LL_o}^{-1} + \frac{\alpha(t)\boldsymbol{\rho}}{N^2}\mathbf{J}\left(I_{2\times 2} + \frac{\alpha(t)\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1}\mathbf{J}^T \right) = \bar{q}\mathbf{P}_{LL_o}^{-1} + \frac{\boldsymbol{\rho}}{N^2}\mathbf{J}\left(I_{2\times 2} + \frac{\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1}\mathbf{J}^T$$

and thus

$$\begin{aligned}
\lim_{t\to\infty}\boldsymbol{\Xi}_{22}(t) &= \mathbf{V}_N\left(\bar{q}\mathbf{V}_N^T\mathbf{P}_{LL_o}^{-1}\mathbf{V}_N + \frac{\boldsymbol{\rho}}{N^2}\mathbf{V}_N^T\mathbf{J}\left(I_{2\times 2} + \frac{\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1}\mathbf{J}^T\mathbf{V}_N \right)^{-1}\mathbf{V}_N \\
&= \mathbf{J}\left(\bar{q}\mathbf{J}^T\mathbf{P}_{LL_o}^{-1}\mathbf{J} + \frac{\boldsymbol{\rho}}{N^2}\mathbf{J}^T\mathbf{J}\left(I_{2\times 2} + \frac{\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1}\mathbf{J}^T\mathbf{J} \right)^{-1}\mathbf{J}
\end{aligned}$$

But $\mathbf{J}^T\mathbf{J} = NI_{2\times 2}$ and thus

$$\begin{aligned}
\lim_{t\to\infty}\boldsymbol{\Xi}_{22}(t) &= \mathbf{J}\left(\bar{q}\mathbf{J}^T\mathbf{P}_{LL_o}^{-1}\mathbf{J} + \boldsymbol{\rho}\left(I_{2\times 2} + \frac{\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1} \right)^{-1}\mathbf{J} \\
&= \mathbf{1}_{N\times N}\otimes\left(\bar{q}\mathbf{J}^T\mathbf{P}_{LL_o}^{-1}\mathbf{J} + \boldsymbol{\rho}\left(I_{2\times 2} + \frac{\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1} \right)^{-1} \\
&= \mathbf{1}_{N\times N}\otimes\left(\bar{q}\mathbf{J}^T\mathbf{P}_{LL_o}^{-1}\mathbf{J} + \boldsymbol{\rho}\left(I_{2\times 2} + \frac{\boldsymbol{\rho}}{\bar{q}}\mathbf{P}_{rr_o}\right)^{-1} \right)^{-1} \\
&= \frac{1}{\bar{q}}\mathbf{1}_{N\times N}\otimes\left(\mathbf{J}^T\mathbf{P}_{LL_o}^{-1}\mathbf{J} + \left(\frac{\bar{q}}{\boldsymbol{\rho}}I_{2\times 2} + \mathbf{P}_{rr_o}\right)^{-1} \right)^{-1} \\
&= \frac{1}{\bar{q}}\mathbf{1}_{N\times N}\otimes\left(\mathbf{J}^T\mathbf{P}_{LL_o}^{-1}\mathbf{J} + \left(\sqrt{\frac{\bar{q}r}{N}}I_{2\times 2} + \mathbf{P}_{rr_o}\right)^{-1} \right)
\end{aligned}$$

Substitution of this result in Eq. (82) and evaluation of the asymptotic value of the matrix $\mathbf{P}_b(t)$ in Eq. (79) yields

$$\lim_{t\to\infty}\mathbf{P}_b(t) = \frac{1}{\bar{q}}\mathbf{1}_{(N+1)\times(N+1)}\otimes\left(\mathbf{J}^T\mathbf{P}_{LL_o}^{-1}\mathbf{J} + \left(\sqrt{\frac{\bar{q}r}{N}}I_{2\times 2} + \mathbf{P}_{rr_o}\right)^{-1} \right)^{-1}$$

The steady state value of $\mathbf{P}_n(t)$ is therefore equal to

$$\begin{aligned}
\mathbf{P}_{n_{ss}} &= \lim_{t\to\infty}\mathbf{P}_n(t) \\
&= \lim_{t\to\infty}\left(\mathbf{P}_a(t) + \mathbf{P}_b(t)\right) \\
&= \begin{bmatrix} \frac{1}{\boldsymbol{\rho}}I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \frac{1}{\bar{q}}\mathbf{1}_{(N+1)\times(N+1)}\otimes\left(\mathbf{J}^T\mathbf{P}_{LL_o}^{-1}\mathbf{J} + \left(\sqrt{\frac{\bar{q}r}{N}}I_{2\times 2} + \mathbf{P}_{rr_o}\right)^{-1} \right)^{-1} \qquad (85)
\end{aligned}$$

Finally, the upper bound on the steady state covariance matrix is given by $\bar{\mathbf{P}}_{ss} = \bar{q}\mathbf{P}_{n_{ss}}$. Therefore, we can state the following lemma:

Lemma 2.2 *For Simultaneous Localization and Mapping (SLAM) in 2D, the upper bound on the steady state covariance matrix of the position estimates is*

$$\bar{\mathbf{P}}_{ss} = \begin{bmatrix} \sqrt{\frac{\bar{q}r}{N}} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{1}_{(N+1)\times(N+1)} \otimes \left(\mathbf{J}^T \mathbf{P}_{LL_o}^{-1} \mathbf{J} + \left(\sqrt{\frac{\bar{q}r}{N}} I_{2\times 2} + \mathbf{P}_{rr_o} \right)^{-1} \right)^{-1} \tag{86}$$

where $\mathbf{P}_{rr_o}$ is the initial covariance of the robot's position estimate, $\mathbf{P}_{LL_o}$ is the initial map covariance matrix, $\mathbf{J} = \mathbf{1}_{N\times 1} \otimes I_{2\times 2}$, and the quantities $\bar{q}$ and r are defined in Eqs. (63) and (68) respectively.

We note at this that by employing the assumption that the position estimates for the robot and the map features are initially uncorrelated, a considerably simpler expression has been derived, compared to the 1D analysis, in which this assumption was not imposed. We now apply this result to two cases of interest:

- **Initially Unknown Map**

In SLAM it is usually assumed that the robot starts operating in a totally unknown area. In such cases, the robot can arbitrarily define the origin of the global coordinate frame, and the initial uncertainty about its position is zero. Since no information about the landmarks' positions is available, the uncertainty about the landmarks' positions is infinite. Setting $\mathbf{P}_{rr_0} = \mathbf{0}_{2\times 2}$ and $\mathbf{P}_{LL_0} = \mu I_{2N\times 2N},\ \ \mu \to \infty$ in Eq. (86) results in the following expression for the upper bound of the positioning uncertainty in SLAM, when the robot maps an initially unknown area:

$$\mathbf{P}_{ss} \preceq \bar{\mathbf{P}}_{ss} = \sqrt{\frac{\bar{q}r}{N}} \begin{bmatrix} 2 & \mathbf{1}_{1\times N} \\ \mathbf{0}_{1N\times 1} & \mathbf{1}_{N\times N} \end{bmatrix} \otimes I_{2\times 2} \tag{87}$$

- **Known Landmark Density**

The expression in Eq. (87) provides an upper bound on the worst-case performance of SLAM, under any possible placement of the landmarks in space. However, when the features of the environment to be treated as landmarks are selected (e.g., visual features, prominent geometric features), it is beneficial to choose them so that they are abundant in the environment and evenly distributed throughout it. This way, a more detailed map of an area can be created. In such cases, the density of landmarks in the environment can be *a priori* modeled, for example, by a uniform probability density function (pdf), and this information can be exploited in order to compute a tighter upper bound for the *expected* steady state covariance of the position estimates. Specifically, assuming the initial covariance matrix of the map (before any observations) is $\mathbf{P}_{LL_0} = \mu I_{2N\times 2N},\ \ \mu \to \infty$, while the robot has perfect knowledge about its position, the covariance matrix right after the first landmark observations, and before the robot moves, will be given by

$$\begin{aligned}
\mathbf{P}(0^+) &= \mathbf{P}(0) - \mathbf{P}(0)\mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}(0) \mathbf{H}_o^T + \mathbf{R}_o(0) \right)^{-1} \mathbf{H}_o \mathbf{P}(0) \\
&= \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mu I_{2N\times 2N} \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{2\times 2N} \\ \mu I_{2N\times 2N} \end{bmatrix} (\mu I_{2N\times 2N} + \mathbf{R}_o(0))^{-1} \begin{bmatrix} \mathbf{0}_{2N\times 2} & \mu \mathbf{I}_{2N\times 2N} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mu \mathbf{I}_{2N\times 2N} - \mu \left(\mu I_{2N\times 2N} + \mathbf{R}_o(0) \right)^{-1} \mu \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \left(\frac{1}{\mu} + \mathbf{R}_o^{-1}(0) \right)^{-1} \end{bmatrix}
\end{aligned}$$

and evaluating the limit as $\mu \to \infty$ yields

$$\mathbf{P}(0^+) = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{R}_o(0) \end{bmatrix} \tag{88}$$

We now employ this matrix as the initial value in SLAM, and therefore the upper bound for the steady state covariance matrix, for a *given* initial position of the robot and landmark placement, is given by

$$\bar{\mathbf{P}}_{ss} = \begin{bmatrix} \sqrt{\frac{\bar{q}r}{N}} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{1}_{(N+1)\times(N+1)} \otimes \left(\mathbf{J}^T \mathbf{R}_o(0)^{-1} \mathbf{J} + \sqrt{\frac{N}{\bar{q}r}} I_{2\times 2} \right)^{-1} \tag{89}$$

In Appendix G it is shown that $\mathbf{R}_o(0)^{-1}$ can be calculated in closed form, and this allows us to evaluate the bound in closed form as well. Use of Eq. (103) yields

$$\mathbf{J}^T R_o(0)^{-1} \mathbf{J} = S_1 + S_2 + S_3$$

where

$$\begin{aligned}
S_1 &= \mathbf{J}^T \frac{1}{\sigma_\rho^2} \begin{bmatrix} \widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \widehat{\Delta p}_N \end{bmatrix} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) \begin{bmatrix} \widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \widehat{\Delta p}_N \end{bmatrix}^T \mathbf{J} \\
&= \mathbf{J}^T \frac{1}{\sigma_\rho^2} \begin{bmatrix} \frac{\widehat{\Delta x}_1 \widehat{\Delta x}_1}{\hat{\rho}_1^2} & \frac{\widehat{\Delta x}_1 \widehat{\Delta y}_1}{\hat{\rho}_1^2} & \dots & 0 & 0 \\ \frac{\widehat{\Delta y}_1 \widehat{\Delta x}_1}{\hat{\rho}_1^2} & \frac{\widehat{\Delta y}_1 \widehat{\Delta y}_1}{\hat{\rho}_1^2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\widehat{\Delta x}_N \widehat{\Delta x}_N}{\hat{\rho}_1^2} & \frac{\widehat{\Delta x}_N \widehat{\Delta y}_N}{\hat{\rho}_1^2} \\ 0 & 0 & \dots & \frac{\widehat{\Delta y}_N \widehat{\Delta x}_N}{\hat{\rho}_1^2} & \frac{\widehat{\Delta y}_N \widehat{\Delta y}_N}{\hat{\rho}_1^2} \end{bmatrix} \mathbf{J} \\
&= \frac{1}{\sigma_\rho^2} \begin{bmatrix} \sum_{i=1}^N \frac{\widehat{\Delta x}_i^{\,2}}{\hat{\rho}_i^2} & \sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\hat{\rho}_i^2} \\ \sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\hat{\rho}_i^2} & \sum_{i=1}^N \frac{\widehat{\Delta y}_i^{\,2}}{\hat{\rho}_i^2} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \mathbf{J}^T \frac{1}{\sigma_\theta^2} \begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^4}\right) \begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix}^T \mathbf{J} \\
&= \mathbf{J}^T \frac{1}{\sigma_\theta^2} \begin{bmatrix} \frac{\widehat{\Delta y}_1 \widehat{\Delta y}_1}{\hat{\rho}_1^4} & -\frac{\widehat{\Delta x}_1 \widehat{\Delta y}_1}{\hat{\rho}_1^4} & \dots & 0 & 0 \\ -\frac{\widehat{\Delta x}_1 \widehat{\Delta y}_1}{\hat{\rho}_1^4} & \frac{\widehat{\Delta x}_1 \widehat{\Delta x}_1}{\hat{\rho}_1^4} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\widehat{\Delta y}_N \widehat{\Delta y}_N}{\hat{\rho}_1^4} & -\frac{\widehat{\Delta x}_N \widehat{\Delta y}_N}{\hat{\rho}_1^4} \\ 0 & 0 & \dots & -\frac{\widehat{\Delta x}_N \widehat{\Delta y}_N}{\hat{\rho}_1^4} & \frac{\widehat{\Delta x}_N \widehat{\Delta x}_N}{\hat{\rho}_1^4} \end{bmatrix} \mathbf{J} \\
&= \frac{1}{\sigma_\theta^2} \begin{bmatrix} \sum_{i=1}^N \frac{\widehat{\Delta y}_i^{\,2}}{\hat{\rho}_i^4} & -\sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\hat{\rho}_i^4} \\ -\sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\hat{\rho}_i^4} & \sum_{i=1}^N \frac{\widehat{\Delta x}_i^{\,2}}{\hat{\rho}_i^4} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
S_3 = -\mathbf{J}^T \frac{1}{\sigma_\eta^2} &\begin{bmatrix} \frac{1}{\hat{\rho}_1^2} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \frac{1}{\hat{\rho}_N^2} J\widehat{\Delta p}_N \end{bmatrix} \mathbf{1}_{N\times N} \begin{bmatrix} \frac{1}{\hat{\rho}_1^2} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \frac{1}{\hat{\rho}_N^2} J\widehat{\Delta p}_N \end{bmatrix}^T \mathbf{J} \\
= -\mathbf{J}^T \frac{1}{\sigma_\eta^2} &\begin{bmatrix} \frac{1}{\hat{\rho}_1^2} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \frac{1}{\hat{\rho}_N^2} J\widehat{\Delta p}_N \end{bmatrix} \mathbf{1}_{N\times 1} \mathbf{1}_{1\times N} \begin{bmatrix} \frac{1}{\hat{\rho}_1^2} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \frac{1}{\hat{\rho}_N^2} J\widehat{\Delta p}_N \end{bmatrix}^T \mathbf{J} \\
= -\frac{1}{\sigma_\eta^2} &\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix}
\end{aligned}$$

where

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{J}^T \frac{1}{\sigma_\eta^2} \begin{bmatrix} \frac{1}{\hat{\rho}_1^2} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \frac{1}{\hat{\rho}_N^2} J\widehat{\Delta p}_N \end{bmatrix} \mathbf{1}_{N\times 1}$$
$$= \begin{bmatrix} -\sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\hat{\rho}_i^2} \\ \sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\hat{\rho}_i^2} \end{bmatrix}$$

and therefore

$$S_3 = -\frac{1}{\sigma_\eta^2} \begin{bmatrix} \left(\sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\hat{\rho}_i^2}\right)^2 & -\sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\hat{\rho}_i^2} \sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\hat{\rho}_i^2} \\ -\sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\hat{\rho}_i^2} \sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\hat{\rho}_i^2} & \left(\sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\hat{\rho}_i^2}\right)^2 \end{bmatrix}$$

We can thus write

$$\mathbf{J}^T \mathbf{R}_o(0)^{-1} \mathbf{J} + \sqrt{\frac{N}{\bar{q}r}} I_{2\times 2} = \frac{1}{\sigma_\rho^2} \begin{bmatrix} \sum_{i=1}^N \frac{\widehat{\Delta x}_i^{\,2}}{\hat{\rho}_i^2} & \sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\hat{\rho}_i^2} \\ \sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\hat{\rho}_i^2} & \sum_{i=1}^N \frac{\widehat{\Delta y}_i^{\,2}}{\hat{\rho}_i^2} \end{bmatrix} + \frac{1}{\sigma_\theta^2} \begin{bmatrix} \sum_{i=1}^N \frac{\widehat{\Delta y}_i^{\,2}}{\hat{\rho}_i^4} & -\sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\hat{\rho}_i^4} \\ -\sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\hat{\rho}_i^4} & \sum_{i=1}^N \frac{\widehat{\Delta x}_i^{\,2}}{\hat{\rho}_i^4} \end{bmatrix}$$
$$- \frac{1}{\sigma_\eta^2} \begin{bmatrix} \left(\sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\hat{\rho}_i^2}\right)^2 & -\sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\hat{\rho}_i^2} \sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\hat{\rho}_i^2} \\ -\sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\hat{\rho}_i^2} \sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\hat{\rho}_i^2} & \left(\sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\hat{\rho}_i^2}\right)^2 \end{bmatrix} + \sqrt{\frac{N}{\bar{q}r}} I_{2\times 2} \tag{90}$$

and the inverse of this matrix is equal to

$$\Theta = \left(\mathbf{J}^T \mathbf{R}_o(0)^{-1} \mathbf{J} + \sqrt{\frac{N}{\bar{q}r}} I_{2\times 2}\right)^{-1} = \frac{1}{\det A} A, \quad \text{with} \quad A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \tag{91}$$

and

$$\alpha = \sum_{i=1}^N \frac{\widehat{\Delta y_i^2}}{\sigma_\rho^2 \hat{\rho}_i^2} + \sum_{i=1}^N \frac{\widehat{\Delta x_i^2}}{\sigma_\theta^2 \hat{\rho}_i^4} - \left(\sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\sigma_\eta \hat{\rho}_i^2}\right)^2 + \sqrt{\frac{N}{\bar{q}r}}$$
$$\beta = -\sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\sigma_\rho^2 \hat{\rho}_i^2} + \sum_{i=1}^N \frac{\widehat{\Delta x}_i \widehat{\Delta y}_i}{\sigma_\theta^2 \hat{\rho}_i^4} - \sum_{i=1}^N \frac{\widehat{\Delta x}_i}{\sigma_\eta \hat{\rho}_i^2} \sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\sigma_\eta \hat{\rho}_i^2}$$
$$\gamma = \sum_{i=1}^N \frac{\widehat{\Delta x}_i^{\,2}}{\sigma_\rho^2 \hat{\rho}_i^2} + \sum_{i=1}^N \frac{\widehat{\Delta y}_i^{\,2}}{\sigma_\theta^2 \hat{\rho}_i^4} - \left(\sum_{i=1}^N \frac{\widehat{\Delta y}_i}{\sigma_\eta \hat{\rho}_i^2}\right)^2 + \sqrt{\frac{N}{\bar{q}r}}$$

We note that the upper bound on the steady state covariance matrix can be computed as a *closed-form* function of the initial position of the robot, and the positions of the landmarks. Our prior knowledge about the density of the landmarks can be incorporated in the analysis by treating the position of the robot and landmarks as *random variables* with a known probability distribution function. In this context $\bar{\mathbf{P}}_{ss}$ is a function of random variables, and therefore its mean can be trivially computed with a Monte Carlo method. We note that $\mathbf{P}_{ss} \preceq \bar{\mathbf{P}}_{ss} \Rightarrow E\{\mathbf{P}_{ss}\} \preceq E\{\bar{\mathbf{P}}_{ss}\}$, which implies that the average value of $\bar{\mathbf{P}}_{ss}$ is an upper bound on the *expected* covariance of the position estimates in SLAM. We can thus state the following lemma:

Lemma 2.3 *The maximum expected steady state covariance of the position estimates in SLAM, when the spatial density of landmarks is described by a known pdf, is given by*

$$E\{\mathbf{P}_{ss}\} \preceq \begin{bmatrix} \sqrt{\frac{qr}{N}} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{1}_{(N+1)\times(N+1)} \otimes E\{\Theta\}$$

where Θ can be computed using Eq. (91).

A Appendix: Taylor Series Expansion of the Hyperbolic Sine and Cosine Functions

The Taylor series expansion of the exponential function is given by:

$$e^{at} = \Sigma_{k=0}^{\infty} \frac{a^k t^k}{k!} = 1 + \frac{at}{1!} + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \frac{a^4 t^4}{4!} + \cdots$$

The above relation, when substituting $-t$ instead of t yields:

$$e^{-at} = \Sigma_{k=0}^{\infty} \frac{a^k (-t)^k}{k!} = 1 - \frac{at}{1!} + \frac{a^2 t^2}{2!} - \frac{a^3 t^3}{3!} + \frac{a^4 t^4}{4!} - \cdots$$

Thus, by subtracting and adding the previous two relations, we get:

$$\frac{e^{at} + e^{-at}}{2} = 1 + \frac{1}{2!} a^2 t^2 + \frac{1}{4!} a^4 t^4 + \cdots$$

and

$$\frac{e^{at} - e^{-at}}{2} = \frac{1}{1!} at + \frac{1}{3!} a^3 t^3 + \frac{1}{5!} a^5 t^5 + \cdots$$

The last two functions are the hyperbolic cosine and sine respectively.

B Proof of Lemma 1.1

We denote the SVD of the $N \times N$ matrix $Y(t)$ as

$$\begin{aligned} Y(t) &= W(t)\Lambda(t)Z^T(t) \\ &= [W_1(t)\;\; W_N(t)] \begin{bmatrix} \Lambda_1(t) & \mathbf{0}r \times p \\ \mathbf{0}r \times p & \mathbf{0}p \times p \end{bmatrix} [Z_1(t)\;\; Z_N(t)]^T \end{aligned}$$

where r is the rank of $Y(t)$, $p = N - r$ $\Lambda_1(t)$ is the diagonal matrix of nonzero singular values of $Y(t)$, the columns of $Z_N(t)$ form a basis for the nullspace of $Y(t)$, and the columns of $W_N(t)$ constitute a basis for the nullspace of $Y(t)^T$. We are also assuming that the limit of $Y(t)$ as $t \to \infty$ exists, and satisfies

$$\begin{aligned} Y(\infty) &= W(\infty)\Lambda(\infty)Z^T(\infty) \\ &= [W_1(\infty)\;\; W_N(\infty)] \begin{bmatrix} \Lambda_1(\infty) & \mathbf{0}r \times p \\ \mathbf{0}r \times p & \mathbf{0}p \times p \end{bmatrix} [Z_1(\infty)\;\; Z_N(\infty)]^T \end{aligned}$$

With this notation we write

$$\begin{aligned} (Y(t)t + I_{N\times N})^{-1} &= (W(t)\Lambda(t)Z^T(t)t + I_{N\times N})^{-1} \\ &= (\Lambda(t)Z^T(t)t + W(t)^T)^{-1} W(t)^T \\ &= Z(t)(\Lambda(t)t + W(t)^T Z(t))^{-1} W(t)^T \\ &= \begin{bmatrix} Z_1(t) & Z_N(t) \end{bmatrix} \left(\begin{bmatrix} \Lambda_1(t)t & \mathbf{0}_{r\times p} \\ \mathbf{0}_{p\times r} & \mathbf{0}_{p\times p} \end{bmatrix} + \begin{bmatrix} W_1(t)^T \\ W_N(t)^T \end{bmatrix} \begin{bmatrix} Z_1(t) & Z_N(t) \end{bmatrix} \right)^{-1} \begin{bmatrix} W_1(t)^T \\ W_N(t)^T \end{bmatrix} \\ &= \begin{bmatrix} Z_1(t) & Z_N(t) \end{bmatrix} \begin{bmatrix} \Lambda_1(t)t + W_1(t)^T Z_1(t) & W_1(t)^T Z_N(t) \\ W_N(t)^T Z_1(t) & W_N(t)^T Z_N(t) \end{bmatrix}^{-1} \begin{bmatrix} W_1(t)^T \\ W_N(t)^T \end{bmatrix} \\ &= \begin{bmatrix} Z_1(t) & Z_N(t) \end{bmatrix} \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} W_1(t)^T \\ W_N(t)^T \end{bmatrix} \end{aligned} \quad (92)$$

Employing the formula for the inversion of a partitioned matrix (cf. Appendix D) yields the following expressions for each of the elements $A_i(t)$, $i = 1, 2, 3, 4$:

$$
\begin{aligned}
A_1(t) &= \left(\Lambda_1(t)t + W_1(t)^T Z_1(t) - W_1(t)^T Z_N(t) \left(W_N(t)^T Z_N(t)\right)^{-1} W_N(t)^T Z_1(t)\right)^{-1} \\
A_2(t) &= -A_1 W_1(t)^T Z_N(t) \left(W_N(t)^T Z_N(t)(t)\right)^{-1} \\
A_3(t) &= -\left(W_N(t)^T Z_N(t)(t)\right)^{-1} W_N(t)^T Z_1(t) A_1(t) \\
A_4(t) &= \left(W_N(t)^T Z_N(t) - W_N(t)^T Z_1(t) \left(\Lambda_1(t)t + W_1(t)^T Z_1(t)\right)^{-1} W_1(t)^T Z_N(t)\right)^{-1}
\end{aligned}
$$

Computation of the limits of these matrices as $t \to \infty$ is now possible. We have

$$
\begin{aligned}
\lim_{t\to\infty} A_1(t) &= \left(\Lambda_1(t)t + W_1(t)^T Z_1(t) - W_1(t)^T Z_N(t) \left(W_N(t)^T Z_N(t)\right)^{-1} W_N(t)^T Z_1(t)\right)^{-1} \\
&= \lim_{t\to\infty} \frac{1}{t} \left(\Lambda_1(t) + \frac{1}{t} W_1(t)^T Z_1(t) - W_1(t)^T Z_N(t) \left(W_N(t)^T Z_N(t)\right)^{-1} W_N(t)^T Z_1(t)\right)^{-1} \\
&= \left(\lim_{t\to\infty} \frac{1}{t}\right) \left(\lim_{t\to\infty} \left(\Lambda_1(t) + \frac{1}{t} W_1(t)^T Z_1(t) - W_1(t)^T Z_N(t) \left(W_N(t)^T Z_N(t)\right)^{-1} W_N(t)^T Z_1(t)\right)^{-1}\right) \\
&= \left(\lim_{t\to\infty} \frac{1}{t}\right) \Lambda_1(\infty)^{-1} \\
&= \mathbf{0}_{r\times r}
\end{aligned}
$$

And therefore we also obtain

$$
\lim_{t\to\infty} A_2(t) = \lim_{t\to\infty} A_3^T(t) = \mathbf{0}_{r\times p}
$$

Finally

$$
\begin{aligned}
\lim_{t\to\infty} \left(\Lambda_1(t)t + W_1(t)^T Z_1(t)\right)^{-1} &= \lim_{t\to\infty} \frac{1}{t} \left(\Lambda_1(t) + \frac{1}{t} W_1(t)^T Z_1(t)\right)^{-1} \\
&= \mathbf{0}_{r\times r}
\end{aligned}
$$

and therefore

$$
\lim_{t\to\infty} A_4(t) = (W_N(\infty)^T Z_N(\infty))^{-1}
$$

Substitution in Eq. (92) yields

$$
\begin{aligned}
\lim_{t\to\infty} (Y(t)t + I_{N\times N})^{-1} &= \begin{bmatrix} Z_1(\infty) & Z_N(\infty) \end{bmatrix} \begin{bmatrix} \mathbf{0}_{r\times r} & \mathbf{0}_{r\times p} \\ \mathbf{0}_{p\times r} & (W_N(\infty)^T Z_N(\infty))^{-1} \end{bmatrix} \begin{bmatrix} W_1(\infty)^T \\ W_N(\infty)^T \end{bmatrix} \\
&= Z_N(\infty)(W_N(\infty)^T Z_N(\infty))^{-1} W_N(t)^T
\end{aligned} \tag{93}
$$

which is the desired result.

C Poof of Lemma 2.1

The proof of this lemma is based on the following theorem (adapted from [4]):

Theorem C.1 *For each covariance matrix P_o, "input" matrix*

$$
E_o = \begin{bmatrix} Q & -F^T \\ -F & -C \end{bmatrix}
$$

and time instant t_o there exists an interval $(t_o - a, t_o + a)$, a neighborhood of P_o and E_o and a unique infinitely Frechet differentiable function $P(P_o, E_o)$, such that

- *P is the unique solution of the Riccati differential equation*

$$\dot{P}(t) = FP(t) + P(t)F^T + Q(t) - P(t)C(t)P(t), \quad P(t_o) = P_o \tag{94}$$

- *For given E_o the function $P(P_o, E)$ is monotonically increasing.*

We now note that from Eq. (61) we obtain

$$\bar{\mathbf{P}}(t) = \bar{\mathbf{P}}(0) + \int_0^t \left(\bar{\mathbf{Q}} - \bar{\mathbf{P}}(t)\mathbf{H}_o^T\bar{\mathbf{R}}^{-1}\mathbf{H}_o\bar{\mathbf{P}}(t)\right) dt \tag{95}$$

But $\bar{\mathbf{P}}(0) = \mathbf{P}(0)$, and additionally $\bar{\mathbf{R}} \succeq \mathbf{R}_o(t)$ and $\bar{\mathbf{Q}} \succeq \mathbf{Q}(t)$ for all $t > 0$. Therefore,

$$\begin{aligned}
\bar{\mathbf{P}}(t) &= \mathbf{P}(0) + \int_0^t \left(\bar{\mathbf{Q}} - \bar{\mathbf{P}}(t)\mathbf{H}_o^T\bar{\mathbf{R}}^{-1}\mathbf{H}_o\bar{\mathbf{P}}(t)\right) dt \\
&\succeq \mathbf{P}(0) + \int_0^t \left(\mathbf{Q}(t) - \bar{\mathbf{P}}(t)\mathbf{H}_o^T\mathbf{R}_o(t)^{-1}\mathbf{H}_o\bar{\mathbf{P}}(t)\right) dt \\
&= \mathbf{P}(0) + \lim_{\Delta t\to 0}\sum_{k=0}^{t/\Delta t}\left(\mathbf{Q}(k\Delta t) - \bar{\mathbf{P}}(k\Delta t)\mathbf{H}_o^T\mathbf{R}_o(k\Delta t)^{-1}\mathbf{H}_o\bar{\mathbf{P}}(k\Delta t)\right)\Delta t
\end{aligned} \tag{96}$$

For sufficiently small Δt, the preceding theorem holds within each of the intervals in the sum of Eq. (96). Applying induction and Theorem C.1 is easy to show that for all $k > 0$,

$$\begin{aligned}
\bar{\mathbf{P}}(t) &\succeq \mathbf{P}(0) + \lim_{\Delta t\to 0}\sum_{k=0}^{t/\Delta t}\left(\mathbf{Q}(k\Delta t) - \mathbf{P}(k\Delta t)\mathbf{H}_o^T\mathbf{R}_o(k\Delta t)^{-1}\mathbf{H}_o\mathbf{P}(k\Delta t)\right)\Delta t \\
&= \mathbf{P}(0) + \lim_{\Delta t\to 0}\sum_{k=0}^{t/\Delta t}\dot{\mathbf{P}}(k\Delta t)\Delta t \\
&= \mathbf{P}(0) + \int_0^t \dot{\mathbf{P}}(t)dt \\
&= \mathbf{P}(0)
\end{aligned}$$

D Inversion of a Partitioned Matrix

Let a $(m+n)\times(m+n)$ matrix K be partitioned as

$$K = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Where the $m \times m$ matrix A and the $n \times n$ matrix D are invertible. Then the inverse matrix of K can be written as

$$\begin{bmatrix} X & Y \\ Z & U \end{bmatrix} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \tag{97}$$

E Continuous Time Riccati Equation

For a linear continuous time system, where the state measurements are available continuously, the state model equations are

$$\dot{x}(t) = F(t)x(t) + B(t)u(t) + G(t)w(t) \tag{98}$$

$$z(t) = H(t)x(t) + n(t) \tag{99}$$

where $u(t)$ is the input to the system, $w(t)$ is the dynamic driving noise process having covariance $Q(t)$, $n(t)$ is the measurement noise process, with covariance $R(t)$, $F(t)$ is the matrix describing the dynamic behavior of the states, $B(t)$ is the matrix describing the affect of the inputs on the states, and $H(t)$ is the measurement matrix.

The continuous time Riccati equation, describing the evolution of the state covariance is

$$\dot{P} = FP + PF^T + GQG^T - PH^T R^{-1} HP \tag{100}$$

where the time indices have been dropped for simplicity.

F Matrix Inversion Lemma

If A is $n \times n$, B is $n \times m$, C is $m \times m$ and D is $m \times n$ then:

$$(A^{-1} + BC^{-1}D)^{-1} = A - AB(DAB + C)^{-1}DA \tag{101}$$

G Calculation of $\mathbf{R}_o^{-1}$

From Eq. (60), it is:

$$\begin{aligned}
\mathbf{R}_o &= \begin{bmatrix} \sigma_\rho^2 I_{2\times 2} + \left(\sigma_\phi^2 + \sigma_\theta^2 - \frac{\sigma_\rho^2}{\hat{\rho}_1^2}\right) J\widehat{\Delta p}_1 \widehat{\Delta p}_1^T J^T & .. & \sigma_\phi^2 J\widehat{\Delta p}_1 \widehat{\Delta p}_N^T J^T \\ \vdots & \ddots & \vdots \\ \sigma_\phi^2 J\widehat{\Delta p}_N \widehat{\Delta p}_1^T J^T & .. & \sigma_\rho^2 I_{2\times 2} + \left(\sigma_\phi^2 + \sigma_\theta^2 - \frac{\sigma_\rho^2}{\hat{\rho}_N^2}\right) J\widehat{\Delta p}_N \widehat{\Delta p}_N^T J^T \end{bmatrix} \\
&= \sigma_\rho^2 I_{2N\times 2N} + D\left(\sigma_\theta^2 I_{N\times N} + \sigma_\phi^2 \mathbf{1}_{N\times N} - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\rho_i^2}\right)\right) D^T \\
&= \sigma_\rho^2 I_{2N\times 2N} + D\left(\Xi - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\rho_i^2}\right)\right) D^T
\end{aligned}$$

where

$$D = \begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix} \quad \text{and} \quad \Xi = \sigma_\theta^2 I_{N\times N} + \sigma_\phi^2 \mathbf{1}_{N\times N}$$

Employing the matrix inversion lemma (cf. Appendix F), the inverse of $\mathbf{R}_o$ can be written as:

$$\begin{aligned}
\mathbf{R}_o^{-1} &= \left(\sigma_\rho^2 I_{2N\times 2N} + D\left(\Xi - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\rho_i^2}\right)\right) D^T\right)^{-1} \\
&= \frac{1}{\sigma_\rho^2} I_{2N\times 2N} - \frac{1}{\sigma_\rho^4} D\left(\left(\Xi - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\rho_i^2}\right)\right)^{-1} + \frac{1}{\sigma_\rho^2} D^T D\right)^{-1} D^T \\
&= \frac{1}{\sigma_\rho^2} I_{2N\times 2N} - \frac{1}{\sigma_\rho^4} D\left(\left(\Xi - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\rho_i^2}\right)\right)^{-1} + \operatorname{diag}\left(\frac{\hat{\rho}_i^2}{\sigma_\rho^2}\right)\right)^{-1} D^T
\end{aligned}$$

where the last line follows from the definition of matrix D. By applying the matrix inversion lemma once more we have

$$\left(-\operatorname{diag}\left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2}\right) + \Xi\right)^{-1} = -\operatorname{diag}\left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2}\right)^{-1} - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2}\right)^{-1}\left(\Xi^{-1} - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2}\right)^{-1}\right)^{-1} \operatorname{diag}\left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2}\right)^{-1}$$

and substitution in Eq. (102) yields

$$\begin{aligned}
\mathbf{R}_o^{-1} &= \frac{1}{\sigma_\rho^2} I_{2N\times 2N} - \frac{1}{\sigma_\rho^4} D \left(-\operatorname{diag}\left(\frac{\sigma_\rho^2}{\rho_i^2}\right)^{-1} \left(\Xi^{-1} - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2}\right)^{-1} \right)^{-1} \operatorname{diag}\left(\frac{\sigma_\rho^2}{\rho_i^2}\right)^{-1} \right)^{-1} D^T \\
&= \frac{1}{\sigma_\rho^2} I_{2N\times 2N} + D \operatorname{diag}\left(\frac{1}{\sigma_\rho^2}\right) \left(\Xi^{-1} - \operatorname{diag}\left(\frac{\sigma_\rho^2}{\hat{\rho}_i^2}\right)^{-1} \right) \operatorname{diag}\left(\frac{1}{\sigma_\rho^2}\right) D^T \\
&= \frac{1}{\sigma_\rho^2} I_{2N\times 2N} - \frac{1}{\sigma_\rho^2} D \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) D^T + D \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) \Xi^{-1} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) D^T
\end{aligned}$$

we note that $J\widehat{\Delta p}_i \widehat{\Delta p}_i^T J^T = \hat{\rho}_i^2 I_{2\times 2} - \widehat{\Delta p}_i \widehat{\Delta p}_i^T$, and therefore the above expression can be written as

$$\begin{aligned}
\mathbf{R}_o^{-1} = \frac{1}{\sigma_\rho^2} &\begin{bmatrix} \widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \widehat{\Delta p}_N \end{bmatrix} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) \begin{bmatrix} \widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \widehat{\Delta p}_N \end{bmatrix}^T \\
+ &\begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) \left(\sigma_\theta^2 I_{N\times N} + \sigma_\phi^2 \mathbf{1}_{N\times N}\right)^{-1} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) \begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix}^T
\end{aligned} \tag{102}$$

But application of the matrix inversion lemma yields

$$\begin{aligned}
\left(\sigma_\theta^2 I_{N\times N} + \sigma_\phi^2 \mathbf{1}_{N\times N}\right)^{-1} &= \left(\sigma_\theta^2 I_{N\times N} + \sigma_\phi^2 \mathbf{1}_{N\times 1}\mathbf{1}_{1\times N}\right)^{-1} \\
&= \frac{1}{\sigma_\theta^2} I_{N\times N} - \frac{1}{\sigma_\theta^4} \mathbf{1}_{N\times 1} \left(\frac{1}{\sigma_\phi^2} + \frac{N}{\sigma_\theta^2} \right)^{-1} \mathbf{1}_{1\times N} \\
&= \frac{1}{\sigma_\theta^2} I_{N\times N} - \frac{1}{\sigma_\eta^2} \mathbf{1}_{N\times N}
\end{aligned}$$

where

$$\sigma_\eta^2 = \frac{\sigma_\theta^4}{\sigma_\phi^2} + N\sigma_\phi^2$$

Hence, $\mathbf{R}_o^{-1}$ can be written in its final form as

$$\begin{aligned}
\mathbf{R}_o^{-1} = \frac{1}{\sigma_\rho^2} &\begin{bmatrix} \widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \widehat{\Delta p}_N \end{bmatrix} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) \begin{bmatrix} \widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & \widehat{\Delta p}_N \end{bmatrix}^T \\
+ \frac{1}{\sigma_\theta^2} &\begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^4}\right) \begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix}^T \\
- \frac{1}{\sigma_\eta^2} &\begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) \mathbf{1}_{N\times N} \operatorname{diag}\left(\frac{1}{\hat{\rho}_i^2}\right) \begin{bmatrix} J\widehat{\Delta p}_1 & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_N \end{bmatrix}^T
\end{aligned} \tag{103}$$

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